Math 400: Actual Midterm exam - 2017

Closed book exam; no calculators. Adequately explain the steps you take.

1. Use separation of variables to solve

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

for $r \geq 1$ and $-\pi \leq \theta \leq \pi$, subject to

$$u(1, \theta) = \begin{cases} -1 & -\pi < \theta < -\pi/2 \\ 1 & -\pi/2 < \theta < \pi/2 \\ -1 & \pi/2 < \theta < \pi \end{cases}, \quad u(1, -\pi/2) = u(1, \pi/2) = 0.$$

Using

$$\frac{1}{2} \ln \left( \frac{1 + \psi}{1 - \psi} \right) = \sum_{n>0, n \text{ odd}} \frac{\psi^n}{n},$$

sum the series for $u(r, \theta)$, and hence write down a compact logarithmic expression for the solution.

2. Starting with Bessel’s equation with $m = 0$, multiply by $y_0'(r)$, to show that

$$\frac{d}{dr} \left[ r^2 (y')^2 \right] + k^2 r^2 \frac{d}{dr} (y^2) = 0$$

has solution $y = J_0(kr)$. Hence evaluate

$$\int_0^1 r[J_0(knr)]^2 dr,$$

where $z = kn$ is the $n^{th}$ zero of $J_0(z)$. Use separation of variables to solve

$$u_t = k_1^2 u + \frac{1}{r}(ru_r)_r + u_{zz}$$

for $r \leq 1$ and $0 < z < \pi$, subject to $u(1, z, t) = u_z(r, 0, t) = u_z(r, \pi, t) = 0$ and $u(r, z, 0) = 1$. Express your result as a sum involving Bessel functions without any integrals. What is the long-time behaviour of the solution?

**Bonus:** By differentiating Bessel’s equation for $m = 0$ show that $J_1(z) = -J_0'(z)$ (using a suitable normalization); substitute this result back into the original ODE to show that $zJ_0(z) = [zJ_1(z)]'$. 

**Helpful information:**

Bessel’s equation is

$$r^2 y'' + ry' + (k^2 r^2 - m^2)y = 0,$$

and has the solution, $y(r) = J_m(kr)$, which is regular at $r = 0$. $J_0(z)$ and $J_1(z)$ satisfy the relations

$$\frac{d}{dz} J_0(z) = -J_1(z), \quad \frac{d}{dz} [zJ_1(z)] = zJ_0(z).$$
Midterm exam - solution

1. We separate variables: \( u(r, \theta) = X(r)Y(\theta) \), giving \( X(r) = r^m \) or \( r^{-m} \) and \( Y(\theta) = \cos m\theta \) or \( \sin m\theta \). Because \( Y(\theta) \) is \( 2\pi \)-periodic, \( m = 0, 1, 2, \ldots \). We discard \( r^m \) as it is diverges for \( r \to \infty \), and \( \sin m\theta \) because the boundary condition demands that \( u(r, \theta) \) is even in \( \theta \). If \( m = 0 \), both \( X \) and \( Y \) are constant (discarding the divergent solution \( \ln r \)). Hence

\[
   u(r, \theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m r^{-m} \cos m\theta,
\]

with

\[
   a_0 = \frac{2}{\pi} \left( \int_0^{\pi/2} d\theta - \int_{\pi/2}^{\pi} d\theta \right) = 0, \quad a_m = \frac{2}{\pi} \left( \int_0^{\pi/2} \cos m\theta \, d\theta - \int_{\pi/2}^{\pi} \cos m\theta \, d\theta \right).
\]

We find \( a_m = 0 \) if \( m \) is even and \( a_m = -4(-1)^{(m+1)/2}/(m\pi) \) if \( m \) is odd. We rewrite this solution as

\[
   u = -\frac{2i}{\pi} \sum_{m>0, m \text{ odd}} \frac{1}{m} [(ir^{-1}e^{i\theta})^m + (ir^{-1}e^{-i\theta})^m]
\]

and use the summation for each term to obtain

\[
   u = -\frac{i}{\pi} \ln \left( \frac{r^2 - 1 + 2ir \cos \theta}{r^2 - 1 - 2ir \cos \theta} \right).
\]

2. We have \( r_y y_r + k^2 r^2 y_{rr} = 0 \) from multiplying Bessel’s equation with \( m = 0 \) by \( y_r \). But \( y y_r = (y^2)_r/2 \) and \( r y_r y_{rr} = (r^2 y^2)_r/2 \), giving the first result. Integrating this equation from \( r = 0 \) to \( r = 1 \), using \( k = k_n \) and \( J_0(k_n) = 0 \), and lastly integrating the second integral by parts gives

\[
   \int_0^1 r[J_0(k_n r)]^2 \, dr = \frac{1}{2}[J_1(k_n)]^2.
\]

Separating variables for the second part, \( u = X(r)T(t) \), we find

\[
   T_t = (k_1^2 - \lambda - m^2)T, \quad X_{rr} + \frac{1}{r}X_r + \lambda X = 0, \quad Z_{zz} + m^2 Z = 0,
\]

for two separation constants \( \lambda \) and \( m^2 \). We find \( Z \propto \cos mz \) with \( m \) an integer or zero. Next, let \( k_1^2 = \lambda \), then \( X \propto J_0(\kappa r) \). Moreover, the boundary condition, \( X(1) = 0 \) implies that \( J_0(k_1) = 0 \), so \( k = k_n \) for \( n = 1, 2, \ldots \). The solution so far is therefore

\[
   u(r, t) = \sum_{n=1}^{\infty} J_0(k_n r) \left[ \frac{1}{2}a_n^0 e^{(k_1^2-k_n^2)t} + \sum_{m=1}^{\infty} a_m^ne^{(k_1^2-k_n^2-m^2)t} \cos mz \right].
\]

We also have the initial condition \( u(r, z, 0) = 1 \), and so

\[
   a_n^m = \frac{2}{\pi} \int_0^\infty \cos mz \, dz \int_0^1 J_0(k_n r) J_0(k_1 r) \, dr = 0 \quad \& \quad a_n^0 = \frac{2}{\pi} \int_0^\infty \int_0^1 [J_0(k_n r)]^2 \, rdz \int_0^1 J_0(k_1 r) \, dr = \frac{4}{k_n J_1(k_n)},
\]

using the results of the first part of the question and the helpful information about the Bessel functions. Finally, since \( k_n \geq k_1, u \to 2J_0(k_1 r)/[k_1 J_1(k_1)] \) for \( t \to \infty \).