Actual Midterm exam - 2018

Closed book exam; no calculators. Adequately explain the steps you take.

1. Use separation of variables to solve

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

outside the unit disk, $r \geq 1$, subject to

$$u(1, \theta) = \begin{cases} 
0 & \theta = 0 \text{ or } \pi \\
\pi/2 & 0 < \theta < \pi \\
-\pi/2 & \pi < \theta < 2\pi 
\end{cases}.$$

Using

$$\frac{1}{2} \ln \left( \frac{1 + \psi}{1 - \psi} \right) = \sum_{n>0, \text{ odd}} \frac{\psi^n}{n},$$

sum the series for $u(r, \theta)$, and hence write down a compact logarithmic expression for the solution.

2. Given that $J_1(z) = -J_0'(z)$, use Bessel’s equation with $m = 0$ to establish that

$$zJ_1(z) = \int_0^z \hat{z}J_0(\hat{z})d\hat{z}, \quad \frac{d}{dz} \left[ z^2(J_1)^2 \right] + z^2 \frac{d}{dz} (J_0)^2 = 0, \quad \int_0^z \hat{z}[J_0(\hat{z})]^2d\hat{z} = \frac{1}{2} z^2 \left[ (J_1)^2 + (J_0)^2 \right].$$

Use separation of variables to solve

$$u_{tt} = \frac{1}{r}(ru_r)_r$$

inside the unit disk $r \leq 1$, subject to $u(1, t) = 0$, $u(r, 0) = 0$ and $u_t(r, 0) = 1$. Express your result as a sum involving Bessel functions without any integrals.

**Helpful information:**

Bessel’s equation is

$$z^2y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y(z) = J_m(z)$, which is regular at $z = 0$. 
Midterm exam - solution

1. We separate variables: \( u(r, \theta) = X(r)Y(\theta) \), giving \( X(r) = r^m \) or \( r^{-m} \) and \( Y(\theta) = \cos m\theta \) or \( \sin m\theta \).

Because \( Y(\theta) \) is \( 2\pi \)-periodic, \( m = 0, 1, 2, \ldots \). We discard \( r^m \) as it is not regular for \( r \to \infty \), and \( \cos m\theta \) because the boundary condition demands that \( u(r, \theta) \) is odd in \( \theta \). Hence

\[
u(r, \theta) = \sum_{m=1}^{\infty} b_m r^{-m} \sin m\theta, \quad b_m = \int_0^{\pi} \sin m\theta d\theta = [1 - (-1)^m]/m
\]

We rewrite this solution as

\[
u = -i \sum_{m>0, m \text{ odd}} \frac{1}{m} [(r^{-1}e^{i\theta})^m - (r^{-1}e^{-i\theta})^m]
\]

and use the summation for each term to obtain

\[
u = -\frac{i}{2} \ln \left( \frac{r^2 - 1 + 2ir \sin \theta}{r^2 - 1 - 2ir \sin \theta} \right).
\]

2. Dividing Bessel’s equation with \( m = 0 \) and \( y = J_0(z) \) by \( z \) and then integrating furnishes

\[
zJ_0' + \int_0^z \hat{z} J_0(\hat{z}) d\hat{z} = 0,
\]

which gives the first result in view of \( J_1 = -J'_0 \). Next we multiply the equation by \( J'_0 \) and use

\[
zJ_0'(zJ'_0) = \frac{1}{2} [z^2(J'_0)^2]' \quad \text{and} \quad z^2J_0J'_0 = \frac{1}{2} z^2([J_0]^2)'
\]

to arrive at the second result. Last, we integrate the second result in \( z \)

\[
0 = z^2(J_1)^2 + \int_0^z \hat{z}^2 \frac{d}{d\hat{z}} [J_0(\hat{z})]^2 d\hat{z} = z^2(J_1)^2 + z^2(J_0)^2 - 2\int_0^z \hat{z} [J_0(\hat{z})]^2 d\hat{z},
\]

giving the third result.

We now separate variables for the PDE, \( u = X(r)T(t) \), finding

\[
T_{tt} = -k^2T, \quad X_{rr} + \frac{1}{r} X_r + k^2X = 0.
\]

Hence \( X = J_0(kr) \). Moreover, the boundary condition, \( X(1) = 0 \) implies that \( J_0(k) = 0 \), so \( k \) must be a zero of \( J_0(z) \). Denote the \( n \)th such zero by \( k_n \). The solution so far is therefore

\[
u(r, t) = \sum_{n=1}^{\infty} [a_n \cos(k_n t) + b_n \sin(k_n t)] J_0(k_n r)
\]

We also have the initial conditions \( u(r, 0) = 0 \) and \( u_t(r, 0) = 1 \), and so \( a_n = 0 \) and

\[
b_n = \frac{\int_0^1 J_0(k_n r) r dr}{k_n \int_0^1 [J_0(k_n r)]^2 r dr} = \frac{2}{k_n^2 J_1(k_n)},
\]

using the results of the first part of the question.