Traffic flow problems

The flow of cars is modelled by the PDE

$$u_t + [uv(u)]_x = 0$$

1. If $v(u) = 1 - u$ and

$$(a) \quad u(x,0) = \begin{cases} 0 & x < 0 \\ u_0 x^2 & 0 \le x \le 1 \\ u_0 & x > 1 \end{cases}$$

$$(b) \quad u(x,0) = u_0 e^{-|x|},$$

where $0 < u_0 < 1$, determine when and where a shock first forms. Sketch a characteristics diagram and the solution up to the development of the shock. Use the equal areas rule to make sketches of the progress of the shock after it forms for each case.

2. Solve the PDE again for $v(u) = 1 - u$ and

$$u(x,0) = u_0 + H(\pi - |x|)a \sin x$$

with $|a| < u_0$, $|a| + u_0 < 1$ and $H(x)$ the step function. Show that when $a > 0$ a single shock forms and if $a < 0$, two shocks form simultaneously, at time $t = 1/(2|a|)$ (in both cases). Where do these shocks form? By fitting shocks to the multi-valued functions in each case, show that, for $t \gg 1$, the maximum value of u is approximately

$$u_0 + \frac{\pi}{2t} \qquad \text{and} \qquad u_0 + \sqrt{\frac{|a|}{t}}$$

for $a > 0$ and $a < 0$, respectively.

3. Now consider $v(u) = (1-u)^2$. Show that $c(u)$ in $u_t + cu_x = 0$ vanishes for $u = \frac{1}{3}$ and has a minimum at $u = \frac{2}{3}$. If

$$u(x,0) = \frac{u_L + u_R e^{x/L}}{1 + e^{x/L}}$$

with $0 < u_R < \frac{1}{3}$ and $\frac{2}{3} < u_L < 1$. Sketch the initial condition and the development of the solution for $t > 0$. How does the solution differ from the solution with $v = 1 - u$? By consider the limit $L \to 0$, show that the car density changes discontinuously from u_L to $1 - \frac{1}{2}u_L$ at a shock which propagates at speed $c(1 - \frac{1}{2}u_L)$.

Some helpful results: The characteristics solution implies $u = f(x_0)$ and $x_0 = x - tc(u)$ if $u(x,0) = f(x)$ denotes the initial condition. For $v = (1-u)^2$, we have $c(u) = \frac{d}{du}[u(1-u)^2] = (1-u)(1-3u)$ When $L \to 0$, $f(x) \to u_L + (u_R - u_L)H(x)$, and the solution for u that bridges between u_L and u_R all originates from near $x_0 = 0$. Thus, $c(u) = x/t$ or $u = \frac{2}{3} \pm \frac{1}{3}\sqrt{1 + \frac{3x}{t}}$. The shock speed is

$$dX/dt = [u^+(1-u^+)^2 - u^-(1-u^-)^2]/(u^+ - u^-) = 1 + (u^+ + u^-)^2 - u^+u^- - 2(u^+ + u^-).$$

See cars1.tif and cars.tif for pictures.

MATH 400 – Sample Final exam problems

The rules for the actual exam: Closed book exam; no calculators. Answer as much as you can; credit awarded for the best three answers. Adequately explain the steps you take. e.g. if you use an expansion formula, say in one sentence why this is possible; if you quote a special function solution to an ODE, say why this is the correct one. Be as explicit as possible in giving your solutions.

1. Using separation of variables, solve the wave equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=u_{tt},$$

inside the unit sphere, $r\le 1$, with the boundary condition,

$$u=0 \quad \text{on} \quad r=1,$$

and initial condition,

$$u_t(r,\theta,0)=0 \qquad u(r,\theta,0)=(5\cos^3\theta-3\cos\theta)g(r).$$

Hint: for the radial part of the problem, the substitution $R(r)=X(r)/\sqrt{r}$, may prove useful, if one sets $u(r,\theta,t)=R(r)Y(\theta)T(t)$.

1*. Solve Laplace's equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0,$$

outside the unit sphere, $r\ge 1$, with the boundary condition,

$$u(1,\theta,\phi)=\cos 3\theta.$$

1.** Solve the heat equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=u_t,$$

inside the unit sphere, $r\le 1$, with the boundary condition,

$$u=0 \quad \text{on} \quad r=1,$$

and initial condition,

$$u(r,\theta,0)=(3\cos^2\theta-1)g(r).$$

Hint: for the radial part of the problem, the substitution $R(r)=X(r)/\sqrt{r}$, may prove useful.

2. Establish that

$$\mathcal{F}^{-1}\{e^{-a|k|}\}=\frac{a}{\pi(a^2+x^2)} \qquad \text{and} \qquad f\circ g=\mathcal{F}^{-1}\{\hat{f}\hat{g}\},$$

where $\mathcal{F}\{f\}=\hat{f}(k)$, $\mathcal{F}\{g\}=\hat{g}(k)$ and $f\circ g$ is a convolution.

Using the Fourier transform, solve the PDE,

$$u_{xx}+u_{yy}=0, \qquad -\infty<x<\infty, \qquad 0\le y<\infty, \qquad u(x,0)=g(x), \qquad u\to 0 \text{ as } y,|x|\to\infty,$$

expressing your solution in terms of a single integral.

2*. Establish the convolution property of the Fourier transform,

$$\int_{-\infty}^{\infty} f(z)g(x-z)dz = \mathcal{F}^{-1}\{\hat{f}(k)\hat{g}(k)\}.$$

Solve the heat equation, $u_t = u_{xx}$, on the infinite line subject to the initial condition, $u(x,0) = f(x)$, expressing your answer as the convolution integral

$$u(x,t) = \int_{-\infty}^{\infty} f(z)G(x-z,t)dz,$$

where you should determine $G(x,t)$. The region $|x| < a$ of an infinite solid is initially at the uniform temperature T_0, the remainder being at zero temperature. By using the solution above, find the temperature at later times in terms of the error function,

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

2.** Using the Fourier transform, solve the elastic wave equation,

$$u_{tt} + u_{xxxx} = 0, \qquad -\infty < x < \infty, \qquad u(x,0) = \delta(x), \qquad u_t(x,0) = 0,$$

showing that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \cos\left(\frac{x^2}{4t} - \frac{\pi}{4}\right).$$

Note that

$$\int_{-\infty}^{\infty} \cos x^2 \, dx = \int_{-\infty}^{\infty} \sin x^2 \, dx = \sqrt{\frac{\pi}{2}}.$$

3. Establish the shift relation,

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$$

for the Laplace transform, where $\overline{f}(s) = \mathcal{L}\{f(t)\}$.

An age-structured model of a population is based on the PDE,

$$u_t + u_a = -\mu(a)u, \qquad 0 \le a, t < \infty,$$

where $u(a,t)$ dictates the number of individuals with age a at time t; the death rate $\mu(a)$ is a prescribed function, and initially, $u(a,0) = 0$. For age $a = 0$, the birth function is

$$u(0,t) = b(t) + \int_0^{\infty} B(a)u(a,t)da,$$

where $b(t)$ is a prescribed creation function, and $B(a)$ is a prescribed reproductivity.

Using the Laplace transform in time, show that

$$u(a,t) = S(a)\mathcal{L}^{-1}\left\{\frac{\overline{b}(s)e^{-sa}}{D(s)}\right\}, \qquad D(s) = 1 - \int_0^{\infty} B(a)S(a)e^{-sa}da,$$

where the "survival function,"

$$S(a) = \exp\left[-\int_0^a \mu(a')da'\right].$$

Find an explicit solution if $B = 0$.

3*. Show that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$, $\mathcal{L}\{t^n\} = n!/s^{n+1}$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use the Laplace transform to solve the PDE,

$$u_t + u_x = u + t^2, \qquad u(0,t) = f(t), \qquad u(x,0) = 1, \qquad 0 \le x < \infty \qquad 0 \le t < \infty.$$

3.** Establish the relations,

$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s) \qquad \text{and} \qquad \mathcal{L}\{e^{-bt}\} = (b+s)^{-1},$$

for the Laplace transform, where $\overline{f}(s) = \mathcal{L}\{f(t)\}$.

The number of lightbulbs generated in a manufacturing process is given by the PDE,

$$u_t + u_a = -\mu u, \qquad 0 \le a, t < \infty,$$

where μ is a constant,

$$u(a,0) = 0 \qquad \text{and} \qquad u(0,t) = 1 + \int_0^\infty R(a)u(a,t)da,$$

with $R(a)$ a prescribed weighting for production adjustment based on the current bulb population. Using the Laplace transform in time, show that

$$u(a,t) = e^{-\mu a}\mathcal{L}^{-1}\left\{\frac{e^{-sa}}{sD(s)}\right\}, \qquad D(s) = 1 - \int_0^\infty R(a)e^{-sa-\mu a}da.$$

Find an explicit solution if R is constant, paying attention to the two cases $\mu \neq R$ and $\mu = R$.

4. Find an implicit algebraic formula for the solution to

$$u_t - uu_x = 0, \qquad u(x,0) = \tanh x.$$

Sketch the characteristic curves on a space-time diagram and show that a shock forms at $(t,x) = (1,0)$. Write this conservation law in integral form and derive a formula determining the speed of any shocks. By arguing that $u(-x,t) = -u(x,t)$ for $x \neq 0$, determine what happens to the shock formed in the initial-value problem above for $t > 1$.

4*. For

$$u_t - u^2u_x = 0, \qquad u(x,0) = \sin x.$$

show that an infinite number of shocks form at $t = 1$ and find their positions. Draw the characteristic curves on a space-time diagram.

4.** Using the method of characteristics, solve

$$u_t + (u-1)u_x = e^{-t},$$

with $u(x,0) = 1$ for $x < 0$, $u(x,0) = 1 - x$ for $0 \le x \le 1$, and $u(x,0) = 0$ for $x > 1$. Sketch the characteristic curves on a space-time diagram. Show that a shock forms at $t = 1$; at what position does the shock first appear?

Helpful information:

The Sturm-Liouville differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda\sigma(x)y = 0.$$

Legendre's equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0.$$

Bessel's equation is

$$z^2y'' + zy' + (z^2 - m^2)y = 0,$$

and has the solution, $y = J_m(z)$, which is regular at $z = 0$.

Fourier Transforms:

$$\hat{f}(k) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \qquad f(x) = \mathcal{F}^{-1}\{\hat{f}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk$$

Laplace Transform:

$$\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt.$$

Convolution:

$$f \circ g = \int_{-\infty}^{\infty} f(x')g(x-x')dx'.$$

Helpful trigonometric relations:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B, \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B.$$

Some solutions

1. Solved in class.

1*. If $x = \cos\theta$, then $\cos 3\theta = 4x^3 - 3x = \frac{8}{5}P_3(x) - \frac{3}{5}P_1(x)$, using a trig relation and given that $P_1(x) = x$ and $P_3(x)$ is an odd polynomial of degree 3 with normalization $P_n(1) = 1$ (so setting $P_3 = Ax^3 + (1-A)x$ and plugging into Legendre's equation gives $A = \frac{5}{2}$).

The boundary condition is independent of ϕ and so $u = R(r)Y(x)$. The usual separation of variables procedure implies that $R(r)$ is given by r^{-n-1} and $Y(x)$ by $P_n(x)$ where $n = 0, 1, \ldots$ (discarding the other solutions which are irregular for $r \to \infty$ and $x = \pm 1$).

Hence $u = \sum_{n=0}^{\infty} c_n r^{-n-1} P_n(x) = \frac{8}{5}r^{-4}P_3(x) - \frac{3}{5}r^{-2}P_1(x)$, given the boundary condition at $r = 1$.

2.** Fourier transforming the PDE and initial conditions implies $\hat{u}_{tt} = -k^4\hat{u}$, $\hat{u}(k,0) = 1$ and $\hat{u}_t(k,0) = 0$. Hence $\hat{u}(k,t) = \cos k^2 t$. Inverting the Fourier transform gives

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos k^2 t \, e^{ikx} dk = \frac{1}{4\pi}\int_{-\infty}^{\infty} [\cos(k^2t + kx) + i\sin(k^2t + kx) + \cos(k^2t - kx) - i\sin(k^2t - kx)]dk.$$

Using the handy changes of variable $\kappa = \sqrt{t}(k \pm \frac{x}{2t})$ and the trig relations again, one obtains the desired answer given integrals provided.

3*. Laplace transforming the PDE and boundary condition:

$$\bar{u}_x + (s-1)\bar{u} = 1 + \frac{2}{s^3} \qquad \& \qquad \bar{u}(0,s) = \bar{f}(s), \qquad \longrightarrow \qquad \bar{u} = \frac{s^3+2}{s^3(s-1)}[1 - e^{-(s-1)x}] + e^{-(s-1)x}\bar{f}(s).$$

Inverting the Laplace transform after using a partial fraction and the shifting theorem gives

$$u = e^x f(t-x)H(t-x) + 3e^t - 2 - 2t - t^2 - e^x[3e^{t-x} - 2 - 2(t-x) - (t-x)^2]H(t-x).$$

3.** Solved in class.

4. Applying the method of characteristics gives the implicit solution $u = \tanh x_0 = \tanh(x + ut)$. Hence

$$u_x = \frac{\operatorname{sech}^2 x_0}{1 - t\operatorname{sech}^2 x_0}.$$

This first diverges for $t = 1$ at $x_0 = x = 0$. Returning to the integral form of the conservation law, one finds that a shock located at $x = X(t)$ travels with speed

$$\frac{dX}{dt} = -\frac{1}{2}(u^+ + u^-)$$

where $u^\pm$ denote the values of u to either side of the shock. Since $u(x,t)$ is odd, $u^- = -u^+$ and so the shock is stationary.

4.** The characteristic equations are

$$\frac{dx}{dt} = u - 1 \qquad \& \qquad \frac{du}{dt} = e^{-t}.$$

Hence, $u = f(x_0) + 1 - e^{-t}$ and $x = x_0 + tf(x_0) + e^{-t} - 1$, if $u(x,0) = f(x)$ denotes the initial condition. That is, $u = 2 - e^{-t}$ for $x_0 = x - t - e^{-t} + 1 < 0$, $u = 1 - e^{-t}$ for $x_0 = x - e^{-t} + 1 > 1$ and

$$u = \frac{1 - x - t + te^{-t}}{1-t}$$

for $0 \le x_0 \le 1$ or $t + e^{-t} - 1 \le x \le e^{-t}$. The central region shrinks to the point $x = e^{-1}$ at $t = 1$, implying where and when the shock forms.

See q4.tif for pictures.