Math 257/316: Assignment 7

Due Nov 21, in class

Tiger Theory: The eminent biologist, Professor Lurv, derives a model for the colour of the coat of a tiger. He describes this in terms of a pigment variable $u(x,t)$, for which $u = 1$ corresponds to orange, $u = 0$ to grey and $u = -1$ to black. The model is the PDE,

$$u_t = \alpha u - u^3 + u_{xx}, \qquad 0 \le x \le 40\pi,$$

$$u(0,t) = u(40\pi,t) = 0, \qquad u(x,0) = 10^{-2}\sin(x/2) + \epsilon(x - 20\pi),$$

where α and ϵ are parameters.

(a) Assume that u is small so that the cubic term u^3 can be neglected, leaving $u_t = \alpha u + u_{xx}$. Solve this simpler PDE by the method of separation of variables using the initial and boundary values above. For $\epsilon = 0$, use your solution of Lurv's model to suggest why tigers have $\alpha = 1$ but mice have $\alpha = -1$. Explain what happens if you try to use separation of variables when u is not small and the cubic term remains in the PDE.

(b) Again take $\epsilon = 0$, but now allow u to be larger and retain the cubic term in the PDE. Write a finite difference scheme to solve the problem numerically. Do this for $N = 200$, $\Delta t = 0.1$, $\alpha = 1$ and $0 \le t \le 100$ to demonstrate how Lurv's model does indeed predict that an orange/black stripe pattern appears By halving the time step and number of spatial grid points, demonstrate explicitly that your solution is accurate. Show that numerical instability appears if the time step is doubled and show how this is consistent with a violation of the stability criterion $\Delta t < (\Delta x)^2/2$.

(c) Take $\epsilon = 2.5 \times 10^{-5}$. What happens now to your numerical solution? What do you conclude about the suitability of Lurv's model for describing tiger stripes?

(d) For $t = 1$, compare your numerical solutions from (b) and (c) with the analytical solution from (a), truncating the Fourier series after a suitable number of terms.

Solution

(a) Separation of variables with $u(x,t) = T(t)X(x)$ furnishes

$$T' = (\alpha - \lambda)T, \qquad X'' = -\lambda X.$$

where λ is a separation constant. Imposing $X(0) = X(40\pi) = 0$ implies that $X = \sin(nx/40)$ and $\lambda = n^2/40^2$. Hence

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{40}\right) \exp\left[\left(\alpha - \frac{n^2}{40^2}\right) t\right].$$

For the initial condition provided,

$$b_n = \frac{2}{40\pi} \int_0^{40\pi} \left[10^{-2}\sin(x/2) + \epsilon(x - 20\pi)\right] \sin\left(\frac{nx}{40}\right) dx.$$

Thus,

$$u(x,t) = 10^{-2} e^{t(\alpha-1/4)} \sin\left(\frac{x}{2}\right) - 40\epsilon \sum_{n=1}^{\infty} \frac{1}{n}[1 + (-1)^n] \sin\left(\frac{nx}{40}\right) \exp\left[\left(\alpha - \frac{n^2}{40^2}\right) t\right].$$

For $\epsilon = 0$, the solution takes the form of a sinusoidal pattern that grows exponentially if $\alpha = 1$, or decays exponentially if $\alpha = -1$. Hence, a spatial pattern appears for $\alpha = 1$, suggesting that u will grow towards ± 1 and stripes might result, as for a tiger. If $\alpha = -1$, $u \to 0$ for $t \to \infty$, and so the coat become grey, as for a mouse. The PDE cannot be separated into terms depending only on either x or t when the cubic term is present, and so the method of separation of variables fails.

(b) Let $u_n^k = u_n(k\Delta t) = u(x_n, k\Delta t)$ be the solution at the n^{th} position on a grid of N points after k timesteps of duration Δt. The discretization of the PDE is

$$u_n^{k+1} = u_n^k + \Delta t\left[u_n^k - (u_n^k)^3 + \frac{(u_{n+1}^k + u_{n-1}^k - 2u_n^k)}{(\Delta x)^2}\right]$$

where $\Delta x = 40\pi/(N+1)$ is the spatial grid interval, $x_n = n\Delta x$, and $u_0^k = u_{N+1}^k = 0$ in view of the boundary conditions. See figure 1 for the numerical solution for $N = 200$ and $\Delta t = 0.1$. For comparison, figure 3 shows solutions with $N = 100$ and $\Delta t = 0.05$, and another one with $\Delta t = 0.2$ that suffers numerical instability. The solutions not suffering this instability agree with one another, demonstrating the fidelity of the computations. For the numerically unstable solution, $(\Delta x)^2/2 = 0.1974$ which is smaller than Δt and violates the stability condition.

(c) See figure 2; with $\epsilon \neq 0$, the stripes start to merge into a wider pattern with a single band of black and one of orange. This is not very tiger-like, and so Lurv's model is not so good.

(d) See figures 1 and 2.

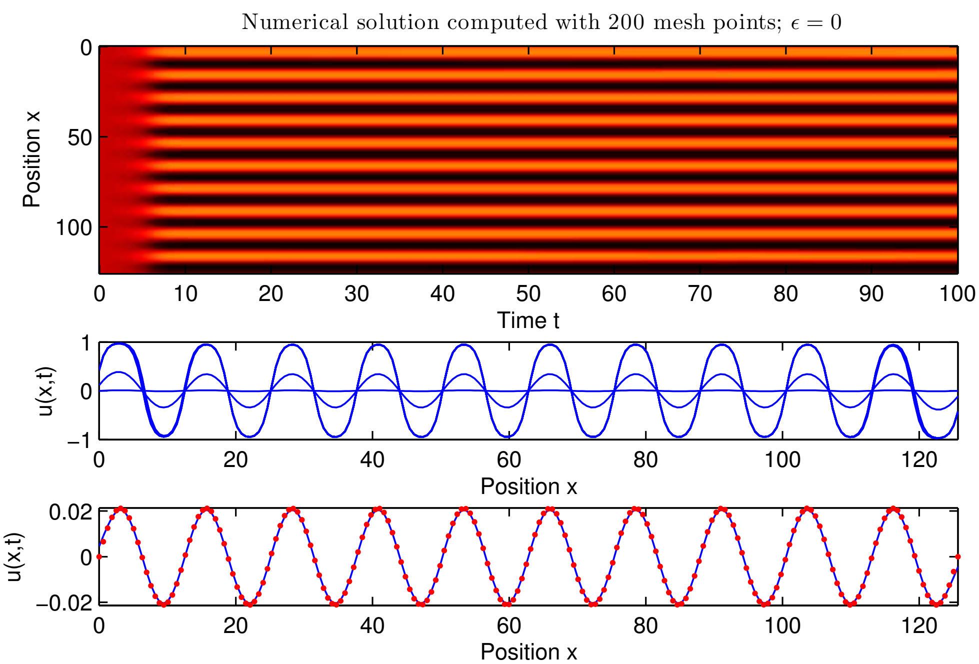

Figure 1: Numerical solutions for $\epsilon = 0$. Top panel: $u(x,t)$ as a colormap on the $(x,t)-$plane. Middle: snapshots of $u(x,t)$ for $t = 0$, 5, 10, ... 100. Bottom: A comparison of the numerical solution at $t = 1$ (solid line) with the separation of variables solution (dots).

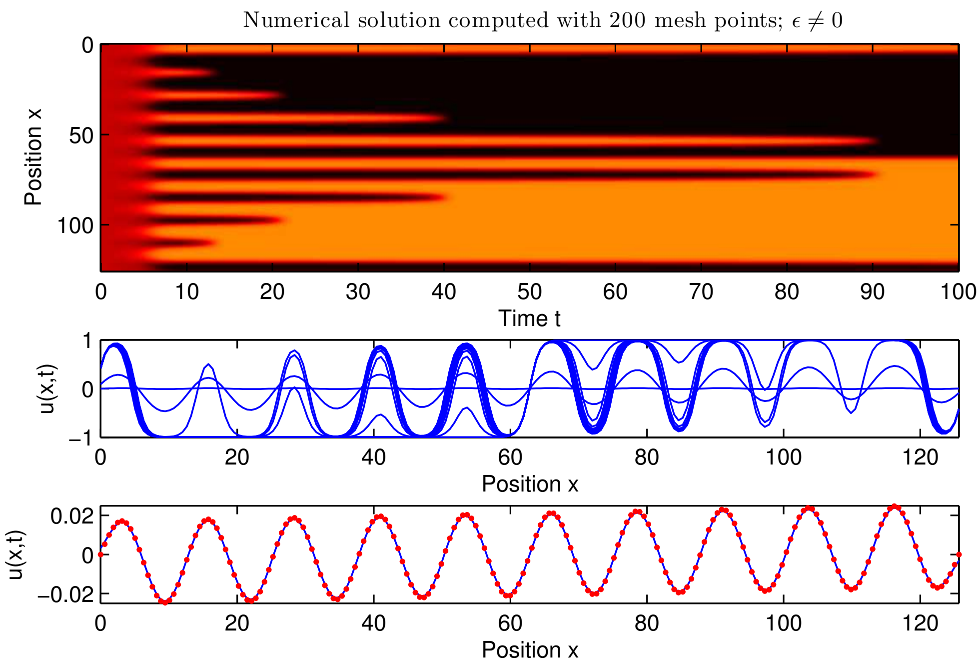

Figure 2: Numerical solutions for $\epsilon = 0$. Top panel: $u(x,t)$ as a colormap on the $(x,t)-$plane. Middle: snapshots of $u(x,t)$ for $t = 0,\ 5,\ 10,\ \ldots\ 100$. Bottom: A comparison of the numerical solution at $t = 1$ (solid line) with the separation of variables solution (dots), truncating tha Fourier series after 50 terms.

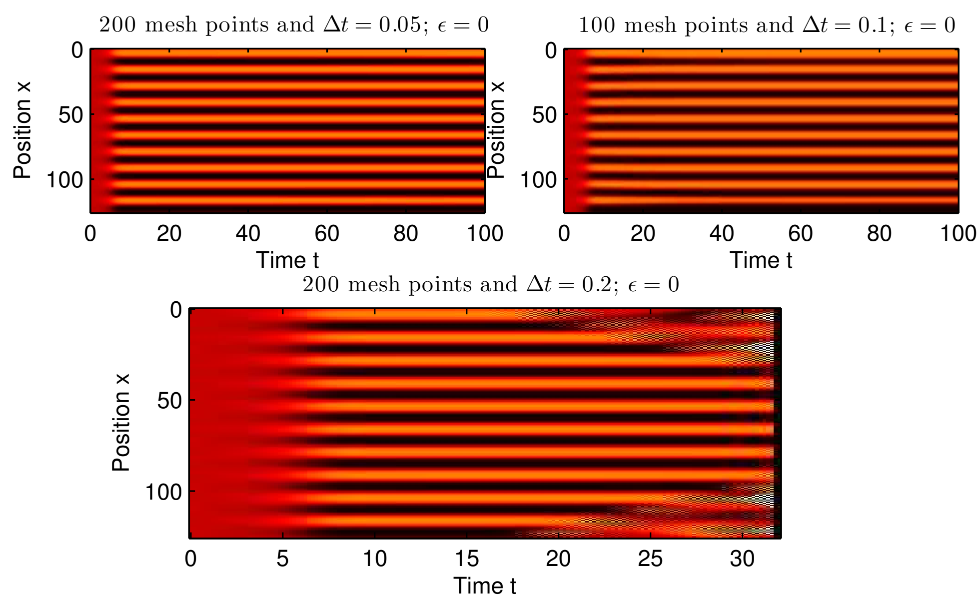

Figure 3: Top: Numerical solutions with half the number of gridpoints in x, and half the time step, Δt. Bottom: numerical instability for $\Delta t = 0.05$ and 200 gridpoints.