Math 257/316: Assignment 6

Due Nov 5, in class

1. Determine d'Alembert's solution to the wave equation with $c = 1$ for the initial conditions,

$$(a)\ u(x,0) = \frac{2}{1+x^2},\ u_t(x,0) = 0, \qquad (b)\ u(x,0) = 0,\ u_t(x,0) = \frac{4x}{(1+x^2)^2}.$$

Plot the solutions for $-40 < x < 40$ at the times $t = [0, 0.2, 0.5, 1, 2, 5, 10, 20, 30]$. *Recommendation: use computer graphics.*

2. Consider the wave equation with unit speed, $c = 1$, and the initial conditions,

$$u(x,0) = 2\chi(x),\ u_t(x,0) = 0,$$

where $\chi(x)$ is the top-hat function:

$$\chi(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

On a space-time diagram (a plot of the $(x,t)-$plane) identify the regions where $u(x,t) \neq 0$; this is the "domain of influence" of the non-zero part of the initial condition. What are the values of $u(x,t)$ over these regions. Sketch the solution for $-3 < x < 3$ and $t = j/4$, $j = 0, 1, 2, ..., 7$.

3. Solve Laplace's equation, $u_{xx} + u_{yy} = 0$ on $0 \le x \le \pi$ and $0 \le y \le \pi$ subject to

$$(a)\ u(x,0) = u(0,y) = u(x,\pi) = 0,\ u(\pi,y) = \pi - |\pi - 2y|$$

$$(b)\ u(x,0) = u(0,y) = u(\pi,y) = 0,\ u(x,\pi) = x(\pi - x)$$

$$(c)\ u(x,0) = u(0,y) = 0,\ u(x,\pi) = x(\pi - x),\ u(\pi,y) = \pi - |\pi - 2y|.$$

Solutions:

1. For the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, d'Alembert's solution is

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\, dz$$

Hence

$$(a)\ u(x,t) = \frac{1}{1+(x-t)^2} + \frac{1}{1+(x+t)^2}, \qquad (b)\ u(x,t) = \frac{1}{1+(x+t)^2} - \frac{1}{1+(x-t)^2}$$

4 points; 8 points for the pictures (4 each).

2(a). The space-time plane contains the regions A, B and C where u is non-zero: A is $t-1 < x < 1-t$. B is $-1+t < x < 1+t$ and $x > 1-t$. C is $-1-t < x < 1-t$ and $x < -1+t$ In A we have $u = 2$; in B and C we have $u = 1$. **8 points.**

3. In view of the zero boundary conditions, separation of variables indicates that

$$(a)\ u = \sum_{n=1}^{\infty} b_n \sin ny \sinh nx, \qquad (b)\ u = \sum_{n=1}^{\infty} B_n \sinh ny \sin nx.$$

Applying the non-trivial boundary conditions:

$$(a)\ \pi - |\pi - 2y| = \sum_{n=1} b_n \sinh n\pi \sin ny, \qquad (b)\ x(\pi - x) = \sum_{n=1} B_n \sin nx \sinh n\pi,$$

giving

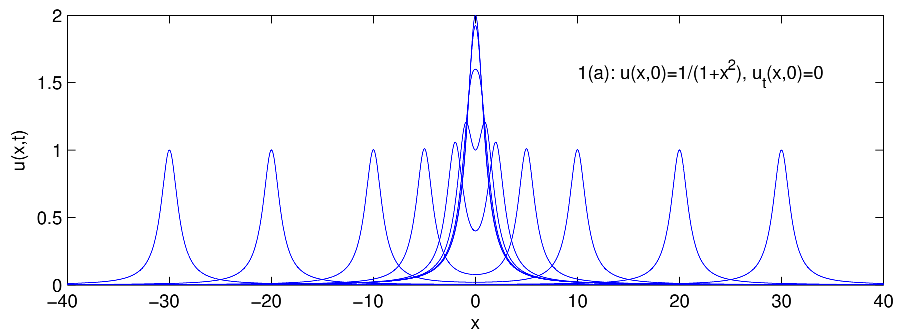
$$b_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} [\pi - |\pi - 2y|] \sin ny dy = \frac{8}{\pi n^2} \sin\frac{n\pi}{2}$$

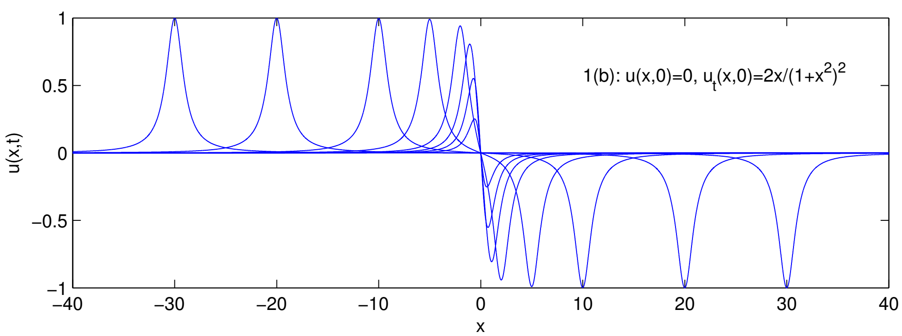
and

$$B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx dx = \frac{4}{\pi n^3}[1 - (-1)^n].$$

8 points (4 points each).

For part (c) we observe that the solution is simply the sum of the solutions to parts (a) and (b) as together they satisfy the needed boudnary conditions. **2 points.**





Qu. 2(a)