Math 257/316: Assignment 5

Due Oct 24, in class

Solve the diffusion problems,

$$(a) \qquad u_t = u_{xx}, \qquad u(x,0) = 0, \qquad u(0,t) = 0 \qquad u(\pi,t) = 1.$$

$$(b) \qquad u_t = u_{xx} + t\sin 2x, \qquad u(x,0) = 0, \qquad u(0,t) = u(\pi,t) = 0.$$

$$(c) \qquad u_t = u_{xx} - \frac{x}{\pi}, \qquad u(x,0) = 0, \qquad u(0,t) = u(\pi,t) = 0.$$

$$(d) \qquad u_t = u_{xx}, \qquad u(x,0) = 0, \qquad u(0,t) = 0 \qquad u(\pi,t) = t.$$

Solutions

(a) Let

$$u(x,t) = U(x,t) + \frac{x}{\pi}$$

Then $U(x,t)$ solves

$$U_t = U_{xx}, \qquad U(x,0) = -\frac{x}{\pi}, \qquad U(0,t) = U(\pi,t) = 0,$$

which has solution

$$U(x,t) = \sum_{n=1} b_n e^{-n^2 t} \sin nx, \qquad b_n = \frac{2}{\pi} \int_0^\pi U(x,0) \sin nx \, dx = \frac{2(-1)^n}{\pi n}.$$

(b) Put

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx.$$

The PDE is

$$\sum_{n=1}^{\infty} (\dot{b_n} + n^2 b_n) \sin nx = t \sin 2x.$$

Given $b_n(0) = 0$ by the initial condition, we may take $b_n(t) = 0$ for $n \neq 2$ and

$$\dot{b}_2 + 4b_2 = t.$$

Hence

$$u(x,t) = \frac{1}{16}(4t - 1 + e^{-4t}) \sin 2x.$$

(c) Again we use a Fourier sin series in x for $u(x,t)$. The PDE becomes

$$\sum_{n=1}^{\infty} (\dot{b_n} + n^2 b_n) \sin nx = -\frac{x}{\pi}.$$

Multiply by $\sin mx$ and integrate, using $\int_0^\pi \sin(nx)\sin(mx)dx = 0$ if $n \neq m$, or $\pi/2$ if $n = m$:

$$\dot{b}_m + m^2 b_m = -\frac{2}{\pi^2} \int_0^\pi x \sin mx \, dx = \frac{2(-1)^m}{\pi m}.$$

Hence, given $b_m(0) = 0$ by the initial condition, $b_m(t) = 2(-1)^m(1 - e^{-m^2 t})/(\pi m^3)$ and

$$u(x,t) = \sum_{n=1} \frac{2(-1)^n}{\pi n^3}(1 - e^{-n^2 t}) \sin nx.$$

(d) Now we set

$$u(x,t) = U(x,t) + \frac{xt}{\pi}$$

to obtain

$$U_t = U_{xx} - \frac{x}{\pi}, \qquad U(0,t) = U(\pi,0) = 0, \qquad U(x,0) = 0.$$

i.e. $U(x,t)$ solves the same problem as in (c), so the solution is

$$u(x,t) = \frac{xt}{\pi} + \sum_{n=1} \frac{2(-1)^n}{\pi n^3}(1 - e^{-n^2 t}) \sin nx.$$