Math 257/316: Midterm Exmination, 9:00-9:48am, October 15th

Closed book exam. Answer both questions. Adequately justify each step you take.

1. Find a lower bound on the radius of convergence of the series solution,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

to the ordinary differential equation,

$$(1-x^2)y'' - 2xy' + \lambda y = 0,$$

where λ is a constant parameter. Find the recurrence relation satisfied by the coefficients, a_n. Show that there is a solution with the form of a polynomial of finite degree for certain values of λ. Give an explicit solution with this form for $\lambda = 12$; what is the other independent solution for this value of λ?

Solution:

The series solution stated is that about the point $x = 0$, which is an ordinary point of the ODE. The ratios of coefficients are $-2x/(1-x^2)$ and $\lambda/(1-x^2)$, implying that $x = \pm 1$ are singular points, which can limit the radius of convergence ρ of the series solution about $x = 0$. Hence $\rho \geq 1$. **2 pt.**

Plugging the series into the ODE:

$$\sum_{n=0}^{\infty} \left\{ n(n-1)a_n x^{n-2} + [\lambda - n(n+1)\ a_n x^n \right\} = 0$$

Replacing n with $m+2$ in the first term and n with m in the second (and adjusting the lower limits of the sums):

$$\sum_{m=0}^{\infty} \left\{ (m+2)(m+1)a_{m+2} + [\lambda - m(m+1)\ a_m \right\} x^m = 0$$

Setting the coefficient of x^m to zero furnishes the recursion relation,

$$(m+1)(m+2)a_{m+2} = -[\lambda - m(m+1)]a_m. \qquad \textbf{4pt}.$$

The two independent solutions of the ODE therefore split up into a series of even powers, with $a_1 = 0$ and $a_0 \neq 0$, and a series of odd powers, with $a_1 \neq 0$ and $a_0 = 0$. If $\lambda = n(n+1)$ for some integer n, then $0 = a_{n+2} = a_{n+4} = a_{n+6} = ...$ Thus, one of the two independent series solutions will terminate at the power n, leaving a finite-degree polynomial; the other solution remains an infinite series. **4 pt.**

For $\lambda = 12$, the odd power series will terminate for $n = 3$. Explicitly, $a_3 = -5a_1/3$, so the polynomial solution is

$$y(x) = a_1 \left(x - \frac{5}{3}x^3 \right). \qquad \textbf{2pt}.$$

The other solution is the (infinite) even-power series with

$$y(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}, \qquad a_{m+2} = -\frac{[12 - m(m+1)]a_m}{(m+1)(m+2)},$$

and beginning with $a_0 \neq 0$. Explicitly,

$$y(x) = a_0 \left(1 - 6x^2 + 3x^4 + ... \right). \qquad \textbf{2pt}.$$

2. Using the method of separation of variables, derive a solution to

$$u_t = \kappa u_{xx}, \qquad 0 \le x \le 2\pi, \qquad u(0,t) = u(2\pi,t) = 0, \qquad u(x,0) = f(x),$$

giving an explicit formula for any coefficients that appear. What theorem provides a mathematical justification for your solution? Give a specific solution for $f(x) = x$.

Solution:

We pose $u(x,t) = X(x)T(t)$. The PDE can be re-arranged into

$$\frac{X''}{X} = \frac{T'}{\kappa T}.$$

That is, a function of x equal to a function of t, which can only be true if both equal a constant. Let this "separation" constant be $-\lambda^2$. Hence,

$$X" + \lambda^2 X = T' + \kappa\lambda^2 T = 0,$$

with solutions

$$X = A\cos\lambda x + B\sin\lambda x \qquad \text{and} \qquad T = Ce^{-\kappa\lambda^2 t},$$

for some constants A, B and C. The boundary conditions in x imply that $X(0) = X(2\pi) = 0$ and so $A = 0$ and $B\sin(2\pi\lambda) = 0$. Hence, $\lambda = n/2$ for $n = 1, 2, \ldots$ (the choices $B = 0$ and $n = 0$ are trivial). **6 pt.**

We then set $BC = b_n$ and formulate a general solution in terms of the sum,

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\kappa n^2 t/4} \sin(nx/2). \qquad \textbf{2pt}.$$

We next apply the initial condition, $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx/2)$$

Multiplying by $\sin(mx/2)$ (with m an integer) and integrating from $x = 0$ to 2π, then using

$$\int_{-L}^{L} \sin(nx/2)\sin(mx/2)dx = 2\int_0^L \sin(nx/2)\sin(mx/2)dx = \begin{cases} L \text{ if } n = m, \\ 0 \text{ otherwise,} \end{cases}$$

implies that

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx/2)dx.$$

Fourier's theorem, that a periodic function can be expanded in its Fourier series, justifies mathematically that the solution of the PDE can be expressed in this form: if we suitably extend our solution to $-2\pi \le x \le 2\pi$ and then periodically repeat it, we are dealing with an odd periodic function and may therefore express it in terms of a Fourier sin series. **5 pt.**

For $f(x) = x$, we may integrate by parts to find that $b_m = 4(-1)^{m+1}/m$. Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n}(-1)^{n+1} e^{-\kappa n^2 t/4} \sin(nx/2). \qquad \textbf{3pt}.$$