**Math 257/316: Assignment 4**

**Due Oct 8, in class**

**1.** Using the method of separation of variables, solve the PDEs

$$(a) \qquad u_t = u_{xx}, \qquad u(0,t) = u(\pi,t) = 0,\ u(x,0) = \sin 3x + 2\sin x + 6\sin 8x$$

$$(b) \qquad u_t = u_{xx}, \qquad u(0,t) = u(\pi,t) = 0,\ u(x,0) = x(\pi - x)$$

$$(c) \qquad u_t = u_{xx}, \qquad u(0,t) = 0,\ u_x(L,t) = 0,\ u(x,0) = f(x)$$

$$(d) \qquad u_{tt} = c^2 u_{xx}, \qquad u(0,t) = u(L,t) = 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x)$$

In each case, write the coefficients in the solution either explicitly or in terms of integrals over the initial conditions.

**Solutions**

We substitute $u(x,t) = X(x)T(t)$ into the PDEs, re-arrange, introduce a separation constant $\lambda$, and solve the resulting ODEs for $X(x)$ and $T(t)$, determining the possible values of $\lambda$ along the way.

$(a)$ We find $X = \sin n\pi x$ and $T = e^{-n^2 t}$. The general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx$$

With the stated initial condition, we may take $b_1 = 2$, $b_3 = 1$ and $b_8 = 6$ and all other $b_n = 0$. Hence

$$u(x,t) = e^{-9t} \sin 3x + 2e^{-t} \sin x + 6e^{-64t} \sin 8x$$

$(b)$ We find the same general solution as in $(a)$. We need to impose

$$x(\pi - x) = \sum_{n=1} b_n \sin nx.$$

The coefficient are determined as follows (using integration by parts):

$$b_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \; dx = \frac{4}{\pi n^2} [1 - (-1)^n]$$

$(c)$ We find $X = \sin \sqrt{\lambda_n} x$ and $T = e^{-\lambda_n t}$, where $\lambda_n$ must satisfy $\sqrt{\lambda_n} \cos(\sqrt{\lambda_n} L) = 0$. *i.e.* $\lambda_n = (2n+1)^2 \pi^2 / (4L^2)$. The general solution is

$$u = \sum_{n=1}^{\infty} b_n e^{-\lambda_n t} \sin \frac{(2n+1)\pi x}{2L}$$

Imposing the initial condition gives

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n+1)\pi x}{2L} \; dx$$

$(d)$ We find $X = \sin(n\pi x/L)$ and $T = \sin(n\pi ct/L)$ or $\cos(n\pi ct/L)$. The general solution is

$$u = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( b_n \sin \frac{n\pi ct}{L} + B_n \cos \frac{n\pi ct}{L} \right)$$

The initial conditions require that

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \; dx$$

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} \; dx.$$