Math 257/316: Assignment 3

Due Sep 29, in class

1. For the ODE

$$x^2(1-x)^3 y'' + 5xy' + 4y = 0$$

find all the singular points and classify them as regular or irregular. For each regular singular point $x = x_0$ determine the exponent r of the Frobenius solution, $y = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$.

2. Verify that $x = 0$ is a regular singular point of the ODE

$$9x^2 y'' + 9xy' + (x-1)y = 0$$

Determine the recurrence relation satisfied by the coefficients of the Frobenius solution about the singular point for both independent solutions, and hence determine a_1 and a_2 in terms of a_0 for both these solutions.

3. Using finite differences on a spatial grid of N points, turn the PDE

$$u_t = \sin(2t)\sin\left(\frac{2\pi x}{L}\right) - u + u_{xx} - cu_x, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = 0$$

into a set of coupled ODEs in time for $u_n(t) = u(x_n, t)$, where x_n denotes the position of the n^{th} gridpoint in space. Be specific about the locations of the gridpoints and the ODEs for $n = 1$ and N. Finite difference the time derivative to reduce the ODEs to an algebraic problem for $u_n^k = u(x_n, k\Delta t)$, where Δt is the time step and $k = 1$, 2, ... *Extra credit: with a computer, solve the problem numerically upto $t = 10$ for $L = 10$ and $c = 3$ (provide a plot showing the space-time evolution of the solution).*

Solutions

1. The ratios of the coefficients of the ODE are

$$\frac{5}{x(1-x)^3}, \ \frac{4}{x^2(1-x)^3}$$

Thus, $x = 0$ and $x = 1$ are singular points.

We now consider the scaled ratios

$$x\frac{5}{x(1-x)^3}, \ x^2\frac{4}{x^2(1-x)^3} \qquad \text{and} \qquad (x-1)\frac{5}{x(1-x)^3}, \ (x-1)^2\frac{4}{x^2(1-x)^3}$$

The first pair is regular at $x = 0$, indicating that this is a regular singular point; the second pair still diverge at $x = 1$, and so this point is not regular.

To determine r, we consider the first term of the Frobenius solution: $y = a_0(x - x_0)^r + ...$ Plugging this into the ODE and isolating the lowest power of $(x - x_0)$ gives,

$$0 = a_0\left[r(r-1)x^r + 5rx^r + 4\right] + ...$$

That is, $r^2+4r+4 = 0$ or $r = -2$ (implying that the Frobenius solution would contain a logarithm). Alternatively, we may compare the ODE with the Euler equation $x^2y'' + 5xy' + 4y = 0$ and plug in $y = Ax^r$ to find the same equation for r.

2. We introduce the Frobenius solution $y = x^r \sum_{n=0}^{\infty} a_n x^n$ into the ODE:

$$0 = \sum_{n=0}^{\infty}\left[9(r+n)^2 - 1\right] a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}$$

In the first sum, we replace n by $m = n$; in the second. we set $m = n + 1$. This gives

$$0 = \left(9r^2 - 1\right) a_m x^r + \sum_{m=1}^{\infty}\left\{\left[9(r+m)^2 - 1\right] a_m x^{r+m} + a_{m-1}x^{r+m}\right\}$$

Thus, $r = \pm 1/3$ and $a_m = -a_{m-1}/[9m(m + 2r)]$.

$$r = \frac{1}{3}: \ a_1 = -\frac{a_0}{15}, \ a_2 = -\frac{a_1}{48} = \frac{a_0}{720}, \quad r = -\frac{1}{3}: \ a_1 = -\frac{a_0}{3}, \ a_2 = -\frac{a_1}{24} = \frac{a_0}{72}$$

3. Let $\Delta x = 1/(N + 1)$ and set $x_n = n\Delta x$ for $n = 1, 2, ..., N$. Then,

$$\dot{u}_n \approx \sin(2t)\sin\left(\frac{2\pi x_n}{L}\right) - u_n + \frac{(u_{n+1} + u_{n-1} - 2u_n)}{(\Delta x)^2} - (u_{n+1} - u_{n-1})\frac{c}{2\Delta x}.$$

For $n = 1$ and $n = N$, we may set $u_0 = u_{N+1} = 0$ in view of the boundary conditions. Putting $u_t \approx [u(x, t + \Delta t) - u(x, t)]/\Delta t$, and using $u_n^k = u(x_n, k\Delta t)$, we may then write

$$u_n^{k+1} \approx u_n^k + \left[\sin(2k\Delta t)\sin\left(\frac{2\pi x_n}{L}\right)) - u_n^k\right]\Delta t + (u_{n+1}^k + u_{n-1}^k - 2u_n^k)\frac{\Delta t}{(\Delta x)^2} - (u_{n+1}^k - u_{n-1}^k)\frac{c\Delta t}{2\Delta x}.$$