Math 257/316: Assignment 2

Due Sep 19, in class

1. Establish power series, $\sum_{n=0}^{\infty} a_n x^n$, for the solutions to

$$(a) \qquad y'' + (x+2)y' + y = 0,$$

$$(b) \qquad (1-x^2)y'' - xy' + \lambda y = 0,$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$, by computing a_n for $n = 0, 1, \ldots 4$, and providing a recurrence relation for general n. In (b), λ is a constant parameter. For what values of λ can one obtain a polynomial solution (of finite degree) to this ODE? Determine that solution in the case that the polynomial is a quartic.

3. Derive a recurrence relation for the coefficients of the power series solution, $y = \sum_{n=0}^{\infty} a_n x^n$, to

$$(2-x)y'' - xy' - y = 0, \qquad y(0) = 1, \ y'(0) = \frac{1}{2}.$$

Show that $a_m = 2^{-m}$ satisfies this recurrence relation, and hence write down the solution to the ODE is terms of a simple fraction involving $(2-x)$. By substitution into the ODE and initial conditions, demonstrate that your fraction is indeed the solution to the problem.

3. Find the singular points of the following ODEs and classify them as either regular or irregular:

$$(a) \qquad x^2 y'' + (1+3x)y' + y = 0$$

$$(b) \qquad (x^2-4)y'' + (2-x)y' + x^2 y = 0$$

$$(c) \qquad \cos(x)y'' + y' + \cot(x)y = 0$$

4. For the ODEs,

$$(a) \qquad x^2 y'' + y' + \tan(x)y = 0, \quad x_0 = 1$$

$$(b) \qquad x(x^2+1)y'' + x^2 y + \sin(x)y = 0, \quad x_0 = 2$$

establish lower bounds on the radii of convergence about the specified points $x = x_0$.

Solutions

1. Plugging the power series into the ODEs, transforming the integer of the sum to match the powers of x gives

$$(a) \quad (n+2)a_{n+2} + 2a_{n+1} + a_n = 0 \qquad (b) \quad (n+2)(n+1)a_{n+2} = (n^2 - \lambda)a_n$$

with $a_0 = 1$ and $a_1 = 0$ because of the initial conditions. Hence

$$(a) \quad a_2 = -\frac{1}{2},\ a_3 = \frac{1}{3},\ a_4 = -\frac{1}{24} \qquad (b) \quad a_2 = -\frac{\lambda}{2},\ a_3 = 0,\ a_4 = \frac{(4-\lambda)\lambda}{24},$$

For (b), if λ is the square of an even integer, $\lambda = 2m$, then $a_{2m+2} = 0$, so the series terminates to furnish a polynomial. When $\lambda = 16$, $y(x) = 1 - 8x^2 + 8x^4$.

2. The recurrence relation is

$$2(n+2)a_{n+2} - na_{n+1} - a_n = 0$$

with $a_0 = 1$ and $a_1 = 1/2$ according to the initial conditions. Plugging $a_m = 2^{-m}$ into the recurrence relation and starting values verifies that this is indeed a solution. Hence

$$y = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{2}{2-x}.$$

Again plugging this into the ODE and initial conditions verifies that this is the solution.

3. For $P(x)y'' + Q(x)y' + R(x)y = 0$, there are singular points at the locations where Q/P and R/P diverge. For the three ODEs we have the ratios

$$(a)\ \frac{1+3x}{x^2},\ \frac{1}{x^2}, \qquad \longrightarrow \qquad x = 0 \text{ singular}$$

$$(b)\ -\frac{1}{(x+2)},\ \frac{x^2}{x^2-4}, \qquad \longrightarrow \qquad x = \pm 2 \text{ singular}$$

$$(c)\ \frac{1}{\cos x},\ \frac{1}{\sin x}, \qquad \longrightarrow \qquad x = \frac{n\pi}{2} \text{ singular for } n = 0, \pm 1, \pm 2, ...$$

If $x = x_*$ is the singular point and $(x - x_*)Q/P$ and $(x - x_*)^2 R/P$ have finite limits for $x \to x_*$, the point is regular. In (a), the point is not regular; in (b) and (c) all the points are regular.

4. The ratios of coefficients are

$$(a)\ \frac{1}{x^2},\ \frac{\tan x}{x} \qquad \longrightarrow \qquad x = 0 \text{ and } x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \text{ singular.}$$

$$(b)\ \frac{0}{x(x^2+1)},\ \frac{x^2 + \sin x}{x(x^2+1)} \qquad \longrightarrow \qquad x = \pm i \text{ singular.}$$

For (a) and $x_0 = 1$, $x = \pi/2$ is the nearest singularity so the radius convergence $\rho \geq \pi/2 - 1$. In (b) with $x_0 = 2$, $\rho \geq \sqrt{2^2 + 1^2}\sqrt{5}$.