ODEs: one seeks a function of a single variable (e.g. \(y(x)\)) that satisfies a differential equation – a given relation between the function and its derivatives (\(y, y', y'', \ldots\)).

PDEs: one seeks a function of multiple variables (e.g. \(u(x,t)\)) that satisfies a relation between that function and its partial derivatives.

Sample ODE: \(y' = 2y + e^x\)

Sample PDE: \(u_t = u_{xx}\) (the heat or diffusion equation; subscripts used as shorthand for partial derivatives)

An ODE or PDE is LINEAR if the differential equation is a linear combination of the function and its derivatives.

\[
c_0 + c_1 y + c_2 y' = 0;
\]

for an ODE in which \(y\) is related to both its first and second derivative, the equation is linear if

\[
c_0 + c_1 y + c_2 y' + c_3 y'' = 0.
\]

Here, \(c_0, c_1, \ldots\), are either arbitrary functions of \(x\) or constants.

ORDER of an ODE: pick out the highest derivative of \(y(x)\) in the ODE. If \(n\) is the number of derivatives, then the order of the ODE is also \(n\). e.g. \(c_0 + c_1 y + c_2 y' + c_3 y'' = 0\) is both linear and of second order.

Integrating factors

Consider the linear, first-order ODE

\[
y' + p(x)y = q(x)
\]

Let

\[
I(x) = \exp \left( \int pdx \right) \quad \rightarrow \quad \frac{dI}{dx} = pI
\]

Multiply the ODE by the “integrating factor” \(I(x)\):

\[
qI = Iy' + (pI)y = Iy' + I'y = \frac{d}{dx}(Iy)
\]

Hence

\[
Iy = \int qI dx + C
\]

where \(C\) is an arbitrary constant of integration, and so

\[
y = \frac{1}{I} \int qI dx + \frac{C}{I}
\]

e.g. \(y' = 2y + e^x\). We have \(I = \exp \left( \int (-2)dx \right) = e^{-2x}\) and \(\int qI dx = \int e^x e^{-2x} dx = -e^{-x}\), so \(y = Ce^{2x} - e^x\).

e.g. \(y' = 2xy\). We have \(I = \exp \left( \int (-2x)dx \right) = e^{-x^2}\) and \(\int qI dx = 0\), so \(y = Ce^{x^2}\).

e.g. \(y' = 2xy^2 + 3x\) is not linear, silly!

N.B. The solution is not unique given that \(C\) is arbitrary!
Separable first-order ODE

A first-order ODE is separable if it can be written in the form

\[ y' = f(x)g(y) \]

i.e. the dependence on \( x \) and \( y \) can be divided up into two factors. The ODE need not be linear; indeed, it is most often nonlinear.

\[ \text{e.g. } y' = 2xy \quad (f = 2x, \ g = y) \text{ or } y' = 2xe^{-y} \quad (f = 2x, \ g = e^{-y}). \]

Solution strategy: rewrite the ODE and then integrate...

\[
f(x) = \frac{y'}{g(y)} \quad \rightarrow \quad \int f(x)\,dx + C = \int \frac{dy}{dx} \frac{dx}{g(y)} = \int \frac{dy}{g(y)},
\]

where \( C \) is another integration constant. At this stage, since \( f(x) \) and \( g(y) \) are known functions with computable integrals, we’re largely done.

\[ \text{e.g. } y' = 2xy \quad (f = 2x, \ g = y). \] So we have \( \log|y| = C + x^2 \). Exponentiating gives \( |y| = e^{C}e^{x^2} \), and so \( y = \pm e^{C}e^{x^2} \). Let \( C = \pm e^{C} \) (another arbitrary constant), giving \( y = Ce^{x^2} \).

\[ \text{e.g. } y' = 2xe^{-y} \quad (f = 2x, \ g = e^{-y}). \] Now, \( e^y = C + x^2 \), and so \( y = \log(C + x^2) \).

Making the solution unique (fixing \( C \))

To fix the arbitrary constant of integration, we need an additional condition - a starting value or initial condition, of the form \( y(a) = y_0 \) for some given \( a \) and \( y_0 \).

\[ \text{e.g. } y' = 2xy \text{ with } y(0) = 1. \] The solution we get by either noting that this ODE is either linear or separable is \( y = Ce^{x^2} \). Putting \( x = 0 \) gives \( 1 = C \), so \( y = e^{x^2} \).

\[ \text{e.g. } y' = 2xe^{-y} \text{ with } y(0) = 0. \] The solution we get from noting that this is a separable ODE is \( y = \log(C + x^2) \). Putting \( x = 0 \) gives \( 0 = \log C \), or \( C = 1 \). Hence \( y = \log(1 + x^2) \).