Math 256. Midterm exam.

No formula sheet, books or calculators!

Name:
Part I

Circle what you think is the correct answer. +3 for a correct answer, -1 for a wrong answer, 0 for no answer.

1. The ODE $y' - y p(x) = 0$, has the solution,

   (a) $CJ$  
   (b) $J + C$  
   (c) $J - C$  
   (d) $C/J$  
   (e) None of the above,

   where $C$ is a constant and $J = \exp[-\int^x p(\bar{x})d\bar{x}]$.

2. The ODE $y' + f(x)/y = 0$, has the solution,

   (a) $\pm \left[C + 2 \int^x f(\bar{x})d\bar{x}\right]^{1/2}$  
   (b) $\pm \left[C - 2 \int^x f(\bar{x})d\bar{x}\right]^{1/2}$  
   (c) $\pm \left[C + \frac{1}{2} \int^x f(\bar{x})d\bar{x}\right]^2$  
   (d) $\pm \left[C - \frac{1}{2} \int^x f(\bar{x})d\bar{x}\right]^2$  
   (e) None of the above,

   where $C$ is a constant.

3. The ODE $y'' - 4y' + 5y = 0$, has the solution,

   (a) $e^{2x}(A \cos x + B \sin x)$  
   (b) $e^{-2x}(A \cos x + B \sin x)$  
   (c) $Ae^{2x}\cos(2x + B)$  
   (d) $Ae^{x}\cos(x + B)$  
   (e) None of the above,

   where $A$ and $B$ are constants.

4. The ODE $y'' + y' + 2y = 4x^2$, has the particular solution,

   (a) $2x^2 - 2x + 1$  
   (b) $2x^2 - 2x - 1$  
   (c) $2x^2 + 2x + 1$  
   (d) $2x^2 + 2x - 1$  
   (e) None of the above.
Part U

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. Define the integrating factor for the first-order ODE, \( y' + yp(x) = q(x) \).
Hence \((1 - x^2)y' - xy = \sqrt{1 - x^2}(1 + x^2)^2 \) with \( y(0) = 0 \).

\[
I = \exp \int p(x) \, dx
\]

\[
\frac{-x}{1-x^2} = p(x) \quad \therefore \quad I = \exp \int \frac{x \, dx}{1-x^2} = \exp \frac{1}{2} \ln(1-x^2) = \frac{\sqrt{1-x^2}}{2}
\]

\[
\therefore \quad (\sqrt{1-x^2} y)' = 1 + 2x^2 + x^4
\]

\[
y = \frac{c}{\sqrt{1-x^2}} + x + \frac{2}{3} x^3 + \frac{1}{5} x^5
\]

\[y(0) = 0 \implies c = 0\]
2. Solve the ODE,

\[ y'' - 4y' + 4y = e^{\lambda x}, \quad y(0) = y'(0) = 0, \]

for (a) \( \lambda \neq 2 \), and (b) \( \lambda = 2 \).

Homog. Sols. \( y = \lambda e^{\lambda x} \)

Aux Eq. \( m^2 - 4m + 4 = 0 \)

\( (m-2)^2 = 0 \)

\[ \therefore y = (A+Bx)e^{2\lambda x} \quad 3 \]

For \( \lambda \neq 2 \), put \( y = ce^{\lambda x} \)

\( \Rightarrow (\lambda^2 - 4\lambda + 4) c = 1 \)

\[ c = \frac{1}{(\lambda - 2)^2} \]

\[ \therefore y = (A+Bx)e^{2\lambda x} + \frac{e^{\lambda x}}{(\lambda - 2)^2} \quad 3 \]

\( y(0) = 0 \Rightarrow A + \frac{1}{(\lambda - 2)^2} = 0 \quad A = -\frac{1}{(\lambda - 2)^2} \)

\( y'(0) = 0 \Rightarrow 2A + B + \frac{\lambda}{(\lambda - 2)^2} = 0 \)

\( \therefore B = -\frac{(\lambda^2 - 2)}{(\lambda - 2)^2} = -\frac{1}{(\lambda - 2)} \)

\[ y = \frac{-[1 + (\lambda^2 - 2)x]}{(\lambda - 2)^2} e^{2\lambda x} + e^{\lambda x} \]
\[ \lambda = 2 \text{ try } y = cx^2 e^{2x} \]

\[
(2c + 8cx + 4c^2)e^{2x} - 4(2ex + 2cxe^2)e^{2x} + 4cx^2 e^{2x} = e^{2x}
\]

\[ \therefore c = \frac{1}{2} \]

\[ \therefore y = (A + Bx + \frac{1}{2}x^2)e^{2x} \]

\[ y(0) = 0 \Rightarrow A = 0 \]

\[ y'(0) = 0 \Rightarrow B = 0 \]

\[ y = \frac{1}{2} x^2 e^{2x} \]