Math 256. Midterm exam.

No formula sheet, books or calculators!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The ODE $y' - yp(x) = 0$, has the solution,

$$(a)\quad CJ \qquad (b)\quad J + C \qquad (c)\quad J - C \qquad (d)\quad C/J \qquad (e)\quad \textit{None of the above},$$

where C is a constant and $J = \exp[-\int^x p(\tilde{x})d\tilde{x}]$.

2. The ODE $y' + f(x)/y = 0$, has the solution,

$$(a)\quad \pm[C + 2\int^x f(\tilde{x})d\tilde{x}]^{1/2} \qquad (b)\quad \pm[C - 2\int^x f(\tilde{x})d\tilde{x}]^{1/2} \qquad (c)\quad \pm[C + \frac{1}{2}\int^x f(\tilde{x})d\tilde{x}]^2$$

$$(d)\quad \pm[C - \frac{1}{2}\int^x f(\tilde{x})d\tilde{x}]^2 \qquad (e)\quad \textit{None of the above},$$

where C is a constant.

3. The ODE $y'' - 4y' + 5y = 0$, has the solution,

$$(a)\quad e^{2x}(A\cos x + B\sin x) \qquad (b)\quad e^{-2x}(A\cos x + B\sin x) \qquad (c)\quad Ae^{2x}\cos(2x + B)$$

$$(d)\quad Ae^{x}\cos(x + B) \qquad (e)\quad \textit{None of the above},$$

where A and B are constants.

4. The ODE $y'' + y' + 2y = 4x^2$, has the particular solution,

$$(a)\quad 2x^2 - 2x + 1 \qquad (b)\quad 2x^2 - 2x - 1 \qquad (c)\quad 2x^2 + 2x + 1$$

$$(d)\quad 2x^2 + 2x - 1 \qquad (e)\quad \textit{None of the above}.$$

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Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. Define the integrating factor for the first-order ODE, $y' + yp(x) = q(x)$.
Hence $(1-x^2)y' - xy = \sqrt{1-x^2}(1+x^2)^2$ with $y(0) = 0$.

$$I = \exp \int p(x)\,dx \qquad 1$$

$$\frac{-x}{1-x^2} = p(x) \quad \therefore \quad I = \exp -\int \frac{x\,dx}{1-x^2} \; 1 = \exp \tfrac{1}{2}\ln(1-x^2) \; 1$$

$$= \sqrt{1-x^2} \; 1 \qquad 3$$

$$\therefore \left(\sqrt{1-x^2}\, y\right)' = 1 + 2x^2 + x^4$$

$$y = \frac{C}{\sqrt{1-x^2}} + \frac{x + \frac{2}{3}x^3 + \frac{1}{5}x^5}{\sqrt{1-x^2}} \qquad 3$$

$$y(0) = 0 \Rightarrow \underline{C = 0}. \qquad 1$$

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2. Solve the ODE,

$$y'' - 4y' + 4y = e^{\lambda x}, \qquad y(0) = y'(0) = 0,$$

for (a) $\lambda \neq 2$, and (b) $\lambda = 2$.

Homog. Sols. $y = Ae^{mx}$

$\xrightarrow{\text{Aux Eq.}}$ $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$\therefore \quad y = (A + Bx)e^{\overset{2}{\cancel{m}}x}$ 3

For $\lambda \neq 2$, put $y = ce^{\lambda x}$

$\Rightarrow \quad (\lambda^2 - 4\lambda + 4)c = 1$

$c = \dfrac{1}{(\lambda-2)^2}$

$\therefore \quad y = (A + Bx)e^{2x} + \dfrac{e^{\lambda x}}{(\lambda-2)^2}$ 3

$y(0) = 0 \Rightarrow A + \dfrac{1}{(\lambda-2)^2} = 0, \quad A = -1/(\lambda-2)^2$

$y'(0) = 0 \Rightarrow 2A + B + \dfrac{\lambda}{(\lambda-2)^2} = 0$

$\therefore \quad B = -\dfrac{(\lambda \mp 2)}{(\lambda-2)^2} = -\dfrac{1}{(\lambda-2)}$

$$y = \frac{-[1 + (\lambda \mp 2)x]e^{2x} + e^{\lambda x}}{(\lambda-2)^2}$$ 2

$\underline{\lambda = 2}$, try $y = cx^2e^{2x}$

$$(2c + \cancel{8cx} + \cancel{4cx^2})e^{2x}$$
$$-4(\cancel{2cx} + \cancel{2cx^2})e^{2x}$$
$$+\cancel{4cx^2}e^{2x} = e^{2x}$$

$$\therefore \quad c = \frac{1}{2}$$

$$\therefore \quad y = \left(A + Bx + \tfrac{1}{2}x^2\right)e^{2x}$$

$y(0) = 0 \Rightarrow A = 0$

$y'(0) = 0 \Rightarrow B = 0$

$$\underline{\underline{y = \tfrac{1}{2}x^2e^{2x}}}$$

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