Complex numbers review problems

1. For which (real) values of \( \alpha \) are the solutions to \( r^2 + \alpha r + 1 = 0 \) real numbers?
   
   Solving the quadratic:
   
   \[
   r = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4})
   \]
   
   The argument of the square root is positive when \( \alpha^2 > 4 \) (i.e. when \( \alpha > 2 \) or \( \alpha < -2 \)), implying real roots for \( r \). When \(-2 < \alpha < 2\) the roots are complex.

2. For which (real) values of \( \beta \) are the solutions to \( r^2 + i\beta r + 1 = 0 \) purely imaginary numbers? And if \( r^2 + i\beta r - 1 = 0 \)?
   
   Solving the quadratic:
   
   \[
   r = \frac{1}{2}(-i\beta \pm \sqrt{-\beta^2 - 4}) = \frac{i}{2}(\pm\sqrt{-\beta^2 + 4} - \beta)
   \]
   
   Since \( \beta^2 + 4 > 0 \), the roots are always purely imaginary. For the second case
   
   \[
   r = \frac{1}{2}(-i\beta \pm \sqrt{4 - \beta^2}).
   \]
   
   The roots have non-zero real part when \( 4 > \beta^2 \) \((-2 < \beta < 2\)) and are purely imaginary otherwise.

3. Write the complex number \((1 + i)/(1 - i)\) in the forms \( x + iy \) and \( re^{i\theta} \).
   
   \[
   \frac{1 + i}{1 - i} = \frac{(1 + i)^2}{(1 + i)(1 - i)} = \frac{1}{2}(1 + 2i - 1) = i
   \]
   
   Hence \( x = 0 \) and \( y = 1 \). We also have that \( r = \sqrt{x^2 + y^2} = 1 \) and \( \theta = \tan^{-1}(y/x) = \pi/2 \).
   
   Or (in a more roundabout and enjoyable way),
   
   \[
   e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = (1 + i)/\sqrt{2} \quad \text{&} \quad e^{-i\pi/4} = \cos(\pi/4) - i\sin(\pi/4) = (1 - i)/\sqrt{2}
   \]
   
   Thus,
   
   \[
   \frac{1 + i}{1 - i} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i.
   \]

   We immediately read off \( x = 0 \) \( y = r = 1 \) and \( \theta = \pi/2 \).

4. Rewrite the function \( c_1e^{(-2+3i)t} + c_2e^{(-2-3i)t} \) in the form \( a_1e^{-\alpha t}\cos(\beta t) + a_2e^{-\alpha t}\sin(\beta t) \). Show that this also equals \( Re^{-\alpha t}\cos(\beta t + \gamma) \) if \( R = \sqrt{a_1^2 + a_2^2} \) and \( \tan \gamma = -a_2/a_1 \).
   
   Euler’s formula is \( e^{i\theta} = \cos \theta + i\sin \theta \). Hence,
   
   \[
   c_1e^{(-2+3i)t} + c_2e^{(-2-3i)t} = (c_1e^{+3it} + c_2e^{-3it})e^{-2t} = [(c_1 + c_2)\cos(3t) + i(c_1 - c_2)\sin(3t)]e^{-2t},
   \]
   
   and so \( a_1 = c_1 + c_2, \) \( a_2 = i(c_1 - c_2) \), \( \alpha = 2 \) and \( \beta = 3 \).
   
   With the definitions of \( R \) and \( \gamma \), we have \( \cos \gamma = a_1/R \) and \( \sin \gamma = -a_2/R \). Hence
   
   \[
   [a_1\cos(\beta t) + a_2\sin(\beta t)]e^{-\alpha t} = \sqrt{a_1^2 + a_2^2} \left[ \frac{a_1\cos(\beta t)}{\sqrt{a_1^2 + a_2^2}} + \frac{a_2\sin(\beta t)}{\sqrt{a_1^2 + a_2^2}} \right] e^{-\alpha t} = R(\cos(\beta t)\cos \gamma - \sin(\beta t)\sin \gamma)e^{-\alpha t}
   \]
   
   which gives the final result on using a trig formula.