Coursework 5: Laplace transform and characteristics problems

(1) Establish that $L\{t^n\} = s^{-n-1}n!$ and $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$. Use a Laplace transform to solve

$$x^2 u_t + u_x = x^2,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$.

(2) Compute $L\{e^{-|t-a|}\}$ for $a > 0$. Use a Laplace transform to solve

$$u_t + cu_x = ce^{-|x-t|} \quad u(0, t) = u(x, 0) = 0 \& u \to 0 \text{ as } x \to \infty,$$

for $c \neq 1$ and $c = 1$.

(3) Using the method of characteristics, solve

$$x^2 u_t + u_x = u^{-2}$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x, 0) = f(x)$.

(4) Use the method of characteristics to solve

$$u_t + x(1 - x)u_x = x,$$

for $-\infty < x < \infty$, subject to $u(x, 0) = 0$.

(5) Re-solve problem (1) using the method of characteristics.
Coursework 5: Sample Laplace transform and characteristics problems

(1) Establish that \( \mathcal{L}\{e^{at}\} = (s - a)^{-1} \). Use a Laplace transform to solve
\[
u_t + xu_x = x^2,
\]
for \( x \geq 0 \), subject to \( u(x,0) = 0 \) and \( u(0,t) = 0 \).

Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as \( \text{Re}(s) > a \)). Laplace transforming the PDE and boundary condition:
\[
s\overline{u} + x\overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0,s) = 0.
\]
Hence \( \overline{u} = x^2 / [s(s + 2)] \) (using an integrating factor of \( x^s \), and then the boundary condition to discard the homogeneous solution). Inverting the transform using a partial fraction gives
\[
u(x,t) = \frac{1}{2} x^2(1 - e^{-2t}).
\]

(2) Establish that \( \mathcal{L}\{\cos at\} = s / (s^2 + a^2) \) and \( \mathcal{L}\{\sin at\} = a / (s^2 + a^2) \). Use a Laplace transform to show that the solution to
\[
u_t + cu_x = \cos \omega t \delta(x-t), \quad u(0,t) = u(x,0) = 0 \quad \& \quad x > 0,
\]
for \( c > 1 \) is
\[
u(x,t) = \frac{\cos[\Omega(x-ct)/c][H(ct-x) - H(t-x)]}{c-1},
\]
where \( \Omega = \omega c / (c - 1) \). Show that \( \nu(x,t) = \omega^{-1} \sin \omega t \delta(t-x) \) for \( c = 1 \).

Solution: Inserting the functions into the definition of the Laplace transform and integrating by parts connects the transforms together and then gives the desired result (as long as \( \text{Re}(s) > 0 \)). Laplace transforming the PDE and boundary condition:
\[
c\overline{u}_x + s\overline{u} = e^{-sx} \cos \omega x, \quad \overline{u}(0,s) = 0.
\]
Hence
\[
\overline{u}(x,s) = \frac{s(e^{-sx/c} - e^{-sx} \cos \omega x) + \Omega e^{-sx} \sin \omega x}{(c - 1)(s^2 + \Omega^2)}.
\]
Inverting the transform and using a trig relation gives the first result. For \( c = 1 \), we find \( \overline{u}(x,s) = \omega^{-1} e^{-sx} \sin \omega x \), and inverting the transform gives the second result.

(3) Re-solve problem (1) using the method of characteristics.

Solution: The characteristic equations are
\[
\frac{dx}{dt} = x \quad \& \quad \frac{du}{dt} = x^2.
\]
Hence, along the characteristic curves that begin at \( x = x_0 \) at \( t = 0 \), where \( u = 0 \),
\[
x = x_0 e^t \quad \& \quad u = \frac{1}{2} x_0^2(e^{2t} - 1),
\]
which give the same solution as in (1) on eliminating \( x_0 \).
(4) Using the method of characteristics, solve

\[ u_t + x^2 u_x = -u^2 \]

for \(-\infty < x < \infty\) and \(t > 0\), subject to \(u(x, 0) = f(x)\).

**Solution:** The characteristic equations are

\[ \frac{dx}{dt} = x^2 \quad \& \quad \frac{du}{dt} = -u^2. \]

Hence, if \(x = x_0\) and \(u = u_0\) at \(t = 0\),

\[ x = \frac{x_0}{1 - x_0 t} \quad \& \quad u = \frac{u_0}{1 + u_0 t}. \]

But \(u(x, 0) = f(x)\) and so

\[ u(x, t) = \frac{f(x/(1 + xt))}{1 + tf(x/(1 + xt))}. \]