Coursework 5: Laplace transform and characteristics problems

(1) Establish that $\mathcal{L}\{t^n\} = s^{-n-1}n!$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\mathcal{F}(s)$. Use a Laplace transform to solve

$$x^2u_t + u_x = x^2,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$.

(2) Compute $\mathcal{L}\{e^{-|t-a|}\}$ for $a > 0$. Use a Laplace transform to solve

$$u_t + cu_x = ce^{-|x-t|} \quad u(0,t) = u(x,0) = 0 \& u \to 0 \text{ as } x \to \infty,$$

for $c \neq 1$ and $c = 1$.

(3) Using the method of characteristics, solve

$$x^2u_t + u_x = u^{-2}$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

(4) Use the method of characteristics to solve

$$u_t + x(1-x)u_x = x,$$

for $-\infty < x < \infty$, subject to $u(x,0) = 0$.

(5) Re-solve problem (1) using the method of characteristics.
Coursework 5: Sample Laplace transform and characteristics problems

(1) Establish that \( \mathcal{L}\{e^{at}\} = (s - a)^{-1} \). Use a Laplace transform to solve
\[ u_t + xu_x = x^2, \]
for \( x \geq 0 \), subject to \( u(x,0) = 0 \) and \( u(0,t) = 0 \).

Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as \( \text{Re}(s) > a \)). Laplace transforming the PDE and boundary condition:
\[ s\pi + x\pi_x = \frac{x^2}{s}, \quad \pi(0,s) = 0. \]
Hence \( \pi = x^2/[s(s + 2)] \) (using an integrating factor of \( x^s \), and then the boundary condition to discard the homogeneous solution). Inverting the transform using a partial fraction gives
\[ u(x,t) = \frac{1}{2}x^2(1 - e^{-2t}). \]

(2) Establish that \( \mathcal{L}\{\cos at\} = s/(s^2 + a^2) \) and \( \mathcal{L}\{\sin at\} = a/(s^2 + a^2) \). Use a Laplace transform to show that the solution to
\[ u_t + cu_x = \cos \omega t \delta(x-t), \quad u(0,t) = u(x,0) = 0 \quad \text{for} \quad x > 0, \]
for \( c > 1 \) is
\[ u(x,t) = \frac{\cos[\Omega(x-ct)/c][H(ct-x) - H(t-x)]}{c-1}, \]
where \( \Omega = \omega c/(c-1) \). Show that \( u(x,t) = \omega^{-1}\sin \omega t \delta(t-x) \) for \( c = 1 \).

Solution: Inserting the functions into the definition of the Laplace transform and integrating by parts connects the transforms together and then gives the desired result (as long as \( \text{Re}(s) > 0 \)). Laplace transforming the PDE and boundary condition:
\[ cu_x + s\pi = e^{-sx} \cos \omega x, \quad \pi(0,s) = 0. \]
Hence
\[ \pi(x,s) = \frac{s(e^{-sx/c} - e^{-sx} \cos \omega x) + \Omega e^{-sx} \sin \omega x}{(c-1)(s^2 + \Omega^2)}. \]
Inverting the transform and using a trig relation gives the first result. For \( c = 1 \), we find \( \pi(x,s) = \omega^{-1}e^{-sx}\sin \omega x \), and inverting the transform gives the second result.

(3) Re-solve problem (1) using the method of characteristics.

Solution: The characteristic equations are
\[ \frac{dx}{dt} = x \quad \& \quad \frac{du}{dt} = x^2. \]
Hence, along the characteristic curves that begin at \( x = x_0 \) at \( t = 0 \), where \( u = 0 \),
\[ x = x_0e^t \quad \& \quad u = \frac{1}{2}x_0^2(e^{2t} - 1), \]
which give the same solution as in (1) on eliminating \( x_0 \).
(4) Using the method of characteristics, solve
\[ u_t + x^2 u_x = -u^2 \]
for \(-\infty < x < \infty\) and \(t > 0\), subject to \(u(x, 0) = f(x)\).

**Solution:** The characteristic equations are
\[
\frac{dx}{dt} = x^2 \quad \& \quad \frac{du}{dt} = -u^2.
\]
Hence, if \(x = x_0\) and \(u = u_0\) at \(t = 0\),
\[
x = \frac{x_0}{1 - x_0 t} \quad \& \quad u = \frac{u_0}{1 + u_0 t}.
\]
But \(u(x, 0) = f(x)\) and so
\[
u(x, t) = \frac{f(x/(1 + xt))}{1 + tf(x/(1 + xt))}.
\]
Coursework 5: Solutions to actual problems

(1) **Solution:** Inserting \(t^n\) into the definition of the Laplace transform and integrating gives the first result (as long as \(\text{Re}(s) > 0\)). Inserting the second function into the definition and then changing the integration variables gives the second. Laplace transforming the PDE and boundary condition:

\[ sx^2 \overline{u} + \overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0, s) = \overline{f}(s). \]

Hence

\[ \overline{u} = [\overline{f}(s) - s^{-2}]e^{-sx^3/3} + s^{-2}. \]

Inverting the transform with the help of the shifting theorem:

\[ u(x, t) = t + \left[x^3/3 - t + f(t - x^3/3)\right] H\left(t - x^3/3\right). \]

(2) We have

\[ \mathcal{L}\{e^{-|t-a|}\} = \int_0^a e^{-st-a+t} dt + \int_a^\infty e^{-st+a-t} dt = \frac{e^{-a} - e^{-sa}}{s - 1} + \frac{e^{-sa}}{1 + s}. \]

Laplace transforming the PDE:

\[ \overline{u}_x + \frac{s}{c} \overline{u} = \frac{e^{-x} - e^{-sx/c}}{s - 1} + \frac{e^{-sx}}{1 + s}. \]

Hence if \(c \neq 1\),

\[ \overline{u}(x, s) = \frac{c(e^{-x} - e^{-sx/c})}{(1 - c)} \left(\frac{1}{s - 1} - \frac{1}{s - c}\right) + \frac{c(e^{-sx} - e^{-sx/c})}{s(1 - c)} \left(\frac{1}{s + 1} - \frac{1}{s - 1}\right) \]

Inverting the transform, and using the shifting theorem gives

\[ u(x, t) = \frac{c}{(1 - c)} \left[e^{t-x} - e^{ct-x} + H(t-x)\left(2 - e^{x-t} - e^{t-x}\right) + H(t-x/c)(e^{ct-x} + e^{x/c-t} - 2)\right]. \]

For \(c = 1\), we have

\[ \overline{u}(x, s) = \frac{e^{-x} - e^{-sx}}{(s - 1)^2} + x e^{-sx} \left(\frac{1}{s + 1} - \frac{1}{s - 1}\right) \]

which gives

\[ u(x, t) = te^{t-x} + (xe^{x-t} - te^{t-x})H(t - x) \]

(3) The characteristic equations are

\[ \frac{dx}{dt} = \frac{1}{x^2} \quad \& \quad \frac{du}{dt} = -\frac{1}{x^2 u^2}. \]

Hence, given that \(x = x_0\) and \(u = f(x_0)\) at \(t = 0\),

\[ x^3 = x_0^3 + 3t \quad \& \quad u^3 = [f(x_0)]^3 + 3(x_0^2 + 3t)^{1/3} - 3x_0. \]

Replacing \(x_0\) by \((x^3 + 3t)^{1/3}\) in the latter gives \(u(x, t)\).

(4) The characteristic equations are

\[ \frac{dx}{dt} = x(1-x) \quad \& \quad \frac{du}{dt} = x. \]
Hence, given that $x = x_0$ and $u = 0$ at $t = 0$,

$$x = \frac{x_0 e^t}{1 - x_0 + x_0 e^t} \quad \text{or} \quad x_0 = \frac{x}{x + (1 - x)e^t} \quad \& \quad u = \log[1 + x_0(e^t - 1)] = \log\left[1 + \frac{x(e^t - 1)}{x + (1 - x)e^t}\right].$$

(5) The characteristic equations are

$$\frac{dx}{dt} = x^{-2} \quad \& \quad \frac{du}{dt} = 1.$$ 

Hence, if $x = x_0$ and $u = 0$ at $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u = t$$

which is the case for $x^3 > 3t$. But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = f(t_0)$, we find instead that

$$x^3 = 3(t - t_0) \quad \& \quad u = f(t_0) + t - t_0 = f(t - x^3/3) + x^3/3.$$
Coursework 5: Laplace transform and characteristics problems 2018

(1) Establish that \( L\{t^n\} = s^{-n-1}n! \) and \( L\{f(t-a)H(t-a)\} = e^{-as}F(s) \). Use a Laplace transform to solve
\[
x^{\alpha}u_t + u_x = x^{1+2\alpha},
\]
for \( x \geq 0 \), subject to \( u(x,0) = 0 \) and \( u(0,t) = 0 \). Consider all possible choices of the (finite) parameter \( \alpha \geq 0 \).

(2) Establish that \( L\{\delta(x-vt)\} = v^{-1}e^{-sx/v} \) for \( v > 0 \). Use a Laplace transform to solve
\[
u_{tt} = u_{xx} + \delta(x-vt), \quad u(0,t) = u(x,0) = u_t(x,0) = 0 \quad \& \quad u \to 0 \text{ as } x \to \infty,
\]
for \( v \neq 1 \) and \( v = 1 \). Plot your solutions at some representative \( t > 0 \).

(3) Using the method of characteristics, solve
\[
 xu_t + tu_x = -tu^2
\]
for \(-\infty < x < \infty \) and \( t > 0 \), subject to \( u(x,0) = f(x) \).

(4) Use the method of characteristics to solve
\[
 u_t + x^{2n}u_x = x^{2m}
\]
for \(-\infty < x < \infty \), subject to \( u(x,0) = 0 \), where \( n \) and \( m \) are positive integers.

(5) Re-solve problem (1) using the method of characteristics.
Coursework 5: Solutions to actual problems

(1) **Solution:** Inserting $t^n$ into the definition of the Laplace transform and integrating gives the first result (as long as $\text{Re}(s) > 0$). Inserting the second function into the definition and then changing the integration variables gives the second. Laplace transforming the PDE and boundary condition:

\[ sx^{\alpha+1}u_x + u_s = \frac{x^{1+2\alpha}}{s}, \quad u(0, s) = 0. \]

Hence

\[ \frac{d}{dx}\{u \exp[sx^{\alpha+1}/(\alpha + 1)]\} = \frac{1}{s}x^{1+2\alpha} \exp[sx^{\alpha+1}/(\alpha + 1)] \]

giving

\[ u = \frac{x^{\alpha+1}}{s^2} - \frac{(\alpha + 1)}{s^3} \{1 - \exp[-sx^{\alpha+1}/(\alpha + 1)]\} \]

Inverting the transform with the help of the shifting theorem:

\[ u(x, t) = tx^{\alpha+1} - \frac{1}{2}(\alpha + 1)t^2 + \frac{1}{2(\alpha + 1)}H(t - \frac{x^{\alpha+1}}{\alpha + 1}). \]

(2) Inserting the function into the definition of the Laplace transform and using the properties of the $\delta-$function gives the desired result. Laplace transforming the PDE:

\[ u_{xx} - s^2u = \frac{1}{v}e^{-sx/v}, \]

Hence if $v \neq 1$,

\[ u(x, s) = \frac{v(e^{-sx} - e^{-sx/v})}{s^2(1 - v^2)}. \]

Inverting the transform gives

\[ u(x, t) = \frac{1}{v^2 - 1}[v(t - x)H(t - x) - (vt - x)H(vt - x)]. \]

At a representative time, the solution for $c > 1$ forms a triangle above $0 < x < t$, with a peak at $x = vt$ and $u = 0$ for $x > t$ if $v < 1$, or above $0 < x < vt$, with a peak at $x = t$ and $u = 0$ for $x > vt$ if $v > 1$. For $v = 1$, we find $u(x, s) = xe^{-sx}/(2s)$, and so $u(x, t) = \frac{1}{2}xH(t - x)$. The solution now is a right-angle triangle above $0 < x < t$.

(3) The characteristic equations are

\[ \frac{dx}{dt} = \frac{t}{x} \quad \& \quad \frac{du}{dt} = -\frac{tu^2}{x}. \]

Hence, given that $x = x_0$ and $u = f(x_0)$ at $t = 0$,

\[ x^2 = x_0^2 + t^2 \quad \& \quad \frac{1}{u} = \frac{1}{f(x_0)} + x_0 \sqrt{1 + \frac{t^2}{x_0^2}} - x_0. \]

Thus

\[ u = \frac{f(x \sqrt{1 + t^2/x^2})}{1 + [x - x \sqrt{1 + t^2/x^2}]f(x \sqrt{1 + t^2/x^2})}. \]
It's a bit of a mystery what happens in $-t < x < t$. Here the characteristics are $x = \pm \sqrt{t^2 - t_0^2}$ and begin on the $t$-axis, where information is not provided. Hence, without a further assumption, we cannot provide the solution here.

(4) The characteristic equations are

$$\frac{dx}{dt} = x^{2n} \quad \& \quad \frac{du}{dt} = x^{2m}.$$ 

Hence, given that $x = x_0$ and $u = 0$ at $t = 0$, and that $2n - 1 \neq 0$ and $2n - 2m - 1 \neq 0$,

$$x^{-(2n-1)} = x_0^{-(2n-1)} - (2n - 1)t \quad \& \quad u = \frac{[x^{-(2n-1)} + (2n - 1)t](2n-2m-1)/(2n-1) - x^{-(2n-2m-1)}}{2n-2m-1}.$$ 

(5) The characteristic equations are

$$\frac{dx}{dt} = x^{-\alpha} \quad \& \quad \frac{du}{dt} = x^{1+\alpha}.$$ 

Hence, if $x = x_0$ and $u = 0$ at $t = 0$,

$$x^{\alpha+1} = x_0^{\alpha+1} + (\alpha + 1)t \quad \& \quad u = x_0^{\alpha+1}t + \frac{1}{2}(\alpha + 1)t^2,$$

which is the case for $x^{\alpha+1} - (\alpha + 1)t > 0$. But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = 0$, we find instead that

$$x^{\alpha+1} = (\alpha + 1)(t - t_0) \quad \& \quad u = \frac{1}{2}(\alpha + 1)(t - t_0)^2,$$

applying for $x^{\alpha+1} - (\alpha + 1)t < 0$. Eliminating either $x_0$ or $t_0$ now gives the earlier result.
(1) Establish that \( \mathcal{L}\{t^n\} = s^{-n-1}n! \). Use a Laplace transform to solve
\[
xu_t + u_x = x^3,
\]
for \( x \geq 0 \), subject to \( u(x, 0) = 0 \) and \( u(0, t) = 0 \).

(2) Establish that \( \mathcal{L}\{\delta(x - vt)\} = v^{-1}e^{-sx/v} \) and \( \mathcal{L}\{f(t - a)H(t - a)\} = e^{-as}f(s) \). Use a Laplace transform to solve
\[
u_{tt} = c^2[u_{xx} + \delta(x - t)], \quad u(0, t) = u(x, 0) = u_t(x, 0) = 0 \quad \& \quad u \to 0 \text{ as } x \to \infty,
\]
for \( c > 1 \) and \( c = 1 \). Plot your solutions at some representative \( t > 0 \).

(3) Re-solve problem (1) using the method of characteristics.

(4) Using the method of characteristics, solve
\[
u_t + t^2u_x = -u^3
\]
for \( -\infty < x < \infty \) and \( t > 0 \), subject to \( u(x, 0) = f(x) \).
Coursework 5: Solutions to actual problems

(1) Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as \( \text{Re}(s) > 0 \)). Laplace transforming the PDE and boundary condition:

\[
sx\bar{u} + \bar{u}_x = \frac{x^3}{s}, \quad \bar{u}(0, s) = 0.
\]

Hence

\[
\bar{u} = \frac{x^2}{s^2} - \frac{2}{s^3} + \frac{2}{s^4}e^{-sx^2/2}
\]

(using an integrating factor of \( e^{sx^2/2} \)). Inverting the transform using a shifting theorem:

\[
u(x, t) = t(x^2 - t) + \frac{1}{4}(2t - x^2)^2H(2t - x^2).
\]

(2) Inserting the functions into the definition of the Laplace transform gives the desired results. Laplace transforming the PDE:

\[
c^2\bar{u}_{xx} - s^2\bar{u} = -c^2e^{-sx},
\]

Hence if \( c \neq 1 \),

\[
\bar{u}(x, s) = \frac{c^2(e^{-sx} - e^{-sx/c})}{s^2(1 - c^2)}.
\]

Inverting the transform gives

\[
u(x, t) = \frac{c}{(1 - c^2)}[c(t - x)H(t - x) - (ct - x)H(ct - x)].
\]

At a representative time, the solution for \( c > 1 \) forms a triangle above \( 0 < x < ct \), with \( u = 0 \) for \( x > ct \). For \( c = 1 \), we find \( \bar{u}(x, s) = xe^{-sx}/(2s) \), and so \( u(x, t) = \frac{1}{2}xH(t - x) \). The solution now is a right-angle triangle above \( 0 < x < t \).

(3) The characteristic equations are

\[
\frac{dx}{dt} = \frac{1}{x} \quad \& \quad \frac{du}{dt} = x^2.
\]

Hence, if the characteristic intersects \( x = x_0 \) and \( u = 0 \) at \( t = 0 \),

\[
x^2 = x_0^2 + 2t \quad \& \quad u = x_0^2t + t^2,
\]

giving \( u(x, t) = tx^2 - t^2 \), as found previously for \( 2t < x^2 \). But if the characteristic leaves \( x = 0 \) at \( t = t_0 \) with \( u = 0 \), we find instead that

\[
x^2 = 2(t - t_0) \quad \& \quad u = (t - t_0)^2 \equiv \frac{x^4}{4},
\]

which is the earlier result for \( 2t > x^2 \).

(4) The characteristic equations are

\[
\frac{dx}{dt} = t^2 \quad \& \quad \frac{du}{dt} = -u^3.
\]

Hence, if \( x = x_0 \) and \( u = u_0 \) at \( t = 0 \),

\[
x = x_0 + \frac{1}{3}t^3 \quad \& \quad u^2 = \frac{u_0^2}{1 + 2u_0^2t}.
\]

But \( u(x, 0) = f(x) \) and so

\[
u(x, t) = \frac{f(x - t^3/3)}{\sqrt{1 + 2t[f(x - t^3/3)]^2}}.
\]