Coursework 5: Laplace transform and characteristics problems

(1) Establish that $\mathcal{L}\{t^n\} = s^{-n-1}n!$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use a Laplace transform to solve

$$x^2 u_t + u_x = x^2,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = f(t)$.

(2) Compute $\mathcal{L}\{e^{-|t-a|}\}$ for $a > 0$. Use a Laplace transform to solve

$$u_t + cu_x = ce^{-|x-t|} \qquad u(0,t) = u(x,0) = 0 \ \& \ u \to 0 \text{ as } x \to \infty,$$

for $c \neq 1$ and $c = 1$.

(3) Using the method of characteristics, solve

$$x^2 u_t + u_x = u^{-2}$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

(4) Use the method of characteristics to solve

$$u_t + x(1-x)u_x = x,$$

for $-\infty < x < \infty$, subject to $u(x,0) = 0$.

(5) Re-solve problem (1) using the method of characteristics.

Coursework 5: Sample Laplace transform and characteristics problems

(1) Establish that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$. Use a Laplace transform to solve

$$u_t + xu_x = x^2,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = 0$.

Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as $\mathrm{Re}(s) > a$). Laplace transforming the PDE and boundary condition:

$$s\overline{u} + x\overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0,s) = 0.$$

Hence $\overline{u} = x^2/[s(s+2)]$ (using an integrating factor of x^s, and then the boundary condition to discard the homogeneous solution). Inverting the transform using a partial fraction gives

$$u(x,t) = \frac{1}{2}x^2(1-e^{-2t}).$$

(2) Establish that $\mathcal{L}\{\cos at\} = s/(s^2+a^2)$ and $\mathcal{L}\{\sin at\} = a/(s^2+a^2)$. Use a Laplace transform to show that the solution to

$$u_t + cu_x = \cos\omega t\ \delta(x-t), \qquad u(0,t) = u(x,0) = 0\ \&\ x > 0,$$

for $c > 1$ is

$$u(x,t) = \frac{\cos[\Omega(x-ct)/c][H(ct-x) - H(t-x)]}{c-1}.$$

where $\Omega = \omega c/(c-1)$. Show that $u(x,t) = \omega^{-1}\sin\omega t\ \delta(t-x)$ for $c = 1$.

Solution: Inserting the functions into the definition of the Laplace transform and integrating by parts connects the transforms together and then gives the desired result (as long as $\mathrm{Re}(s) > 0$). Laplace transforming the PDE and boundary condition:

$$c\overline{u}_x + s\overline{u} = e^{-sx}\cos\omega x, \qquad \overline{u}(0,s) = 0.$$

Hence

$$\overline{u}(x,s) = \frac{s(e^{-sx/c} - e^{-sx}\cos\omega x) + \Omega e^{-sx}\sin\omega x}{(c-1)(s^2+\Omega^2)}.$$

Inverting the transform and using a trig relation gives the first result. For $c = 1$, we find $\overline{u}(x,s) = \omega^{-1}e^{-sx}\sin\omega x$, and inverting the transform gives the second result.

(3) Re-solve problem (1) using the method of characteristics.

Solution: The characteristic equations are

$$\frac{dx}{dt} = x \quad \& \quad \frac{du}{dt} = x^2.$$

Hence, along the characteristic curves that begin at $x = x_0$ at $t = 0$, where $u = 0$,

$$x = x_0 e^t \quad \& \qquad u = \frac{1}{2}x_0^2(e^{2t}-1),$$

which give the same solution as in (1) on eliminating x_0.

(4) Using the method of characteristics, solve

$$u_t + x^2 u_x = -u^2$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

Solution: The characteristic equations are

$$\frac{dx}{dt} = x^2 \quad \& \quad \frac{du}{dt} = -u^2.$$

Hence, if $x = x_0$ and $u = u_0$ at $t = 0$,

$$x = \frac{x_0}{1 - x_0 t} \quad \& \quad u = \frac{u_0}{1 + u_0 t}.$$

But $u(x,0) = f(x)$ and so

$$u(x,t) = \frac{f(x/(1+xt))}{1 + tf(x/(1+xt))}.$$

Coursework 5: Solutions to actual problems

(1) *Solution:* Inserting t^n into the definition of the Laplace transform and integrating gives the first result (as long as $\mathrm{Re}(s) > 0$). Inserting the second function into the definition and then changing the integration variables gives the second. Laplace transforming the PDE and boundary condition:

$$sx^2\overline{u} + \overline{u}_x = \frac{x^2}{s}, \quad \overline{u}(0,s) = \overline{f}(s).$$

Hence

$$\overline{u} = [\overline{f}(s) - s^{-2}]e^{-sx^3/3} + s^{-2}.$$

Inverting the transform with the help of the shifting theorem:

$$u(x,t) = t + \left[x^3/3 - t + f(t - x^3/3)\right] H\left(t - x^3/3\right).$$

(2) We have

$$\mathcal{L}\{e^{-|t-a|}\} = \int_0^a e^{-st-a+t}dt + \int_a^\infty e^{-st+a-t}dt = \frac{e^{-a} - e^{-sa}}{s-1} + \frac{e^{-sa}}{1+s}.$$

Laplace transforming the PDE:

$$\overline{u}_x + \frac{s}{c}\overline{u} = \frac{e^{-x} - e^{-sx}}{s-1} + \frac{e^{-sx}}{1+s}.$$

Hence if $c \neq 1$,

$$\overline{u}(x,s) = \frac{c(e^{-x} - e^{-sx/c})}{(1-c)}\left(\frac{1}{s-1} - \frac{1}{s-c}\right) + \frac{c(e^{-sx} - e^{-sx/c})}{s(1-c)}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

Inverting the transform, and using the shifting theorem gives

$$u(x,t) = \frac{c}{(1-c)}\left[e^{t-x} - e^{ct-x} + H(t-x)(2 - e^{x-t} - e^{t-x}) + H(t - x/c)(e^{ct-x} + e^{x/c-t} - 2)\right].$$

For $c = 1$, we have

$$\overline{u}(x,s) = \frac{e^{-x} - e^{-sx}}{(s-1)^2} + xe^{-sx}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)$$

which gives

$$u(x,t) = te^{t-x} + (xe^{x-t} - te^{t-x})H(t-x)$$

(3) The characteristic equations are

$$\frac{dx}{dt} = \frac{1}{x^2} \quad \& \quad \frac{du}{dt} = -\frac{1}{x^2u^2}.$$

Hence, given that $x = x_0$ and $u = f(x_0)$ at $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u^3 = [f(x_0)]^3 + 3(x_0^2 + 3t)^{1/3} - 3x_0.$$

Replacing x_0 by $(x^3 + 3t)^{1/3}$ in the latter gives $u(x,t)$.

(4) The characteristic equations are

$$\frac{dx}{dt} = x(1-x) \quad \& \quad \frac{du}{dt} = x.$$

Hence, given that $x = x_0$ and $u = 0$ at $t = 0$,

$$x = \frac{x_0 e^t}{1 - x_0 + x_0 e^t} \quad \text{or} \quad x_0 = \frac{x}{x + (1-x)e^t} \quad \& \quad u = \log[1 + x_0(e^t - 1)] = \log\left[1 + \frac{x(e^t - 1)}{x + (1-x)e^t}\right].$$

(5) The characteristic equations are

$$\frac{dx}{dt} = x^{-2} \quad \& \quad \frac{du}{dt} = 1.$$

Hence, if $x = x_0$ and $u = 0$ at $t = 0$,

$$x^3 = x_0^3 + 3t \quad \& \quad u = t$$

which is the case for $x^3 > 3t$. But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = f(t_0)$, we find instead that

$$x^3 = 3(t - t_0) \quad \& \quad u = f(t_0) + t - t_0 = f(t - x^3/3) + x^3/3.$$

Coursework 5: Laplace transform and characteristics problems 2018

(1) Establish that $\mathcal{L}\{t^n\} = s^{-n-1}n!$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use a Laplace transform to solve

$$x^\alpha u_t + u_x = x^{1+2\alpha},$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = 0$. Consider all possible choices of the (finite) parameter $\alpha \geq 0$.

(2) Establish that $\mathcal{L}\{\delta(x-vt)\} = v^{-1}e^{-sx/v}$ for $v > 0$. Use a Laplace transform to solve

$$u_{tt} = u_{xx} + \delta(x-vt), \qquad u(0,t) = u(x,0) = u_t(x,0) = 0 \ \& \ u \to 0 \text{ as } x \to \infty,$$

for $v \neq 1$ and $v = 1$. Plot your solutions at some representative $t > 0$.

(3) Using the method of characteristics, solve

$$xu_t + tu_x = -tu^2$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

(4) Use the method of characteristics to solve

$$u_t + x^{2n}u_x = x^{2m},$$

for $-\infty < x < \infty$, subject to $u(x,0) = 0$, where n and m are positive integers.

(5) Re-solve problem (1) using the method of characteristics.

Coursework 5: Solutions to actual problems

(1) *Solution:* Inserting t^n into the definition of the Laplace transform and integrating gives the first result (as long as $\mathrm{Re}(s) > 0$). Inserting the second function into the definition and then changing the integration variables gives the second. Laplace transforming the PDE and boundary condition:

$$sx^\alpha \overline{u} + \overline{u}_x = \frac{x^{1+2\alpha}}{s}, \quad \overline{u}(0,s) = 0.$$

Hence

$$\frac{d}{dx}\{\overline{u}\exp[sx^{\alpha+1}/(\alpha+1)]\} = \frac{1}{s}x^{1+2\alpha}\exp[sx^{\alpha+1}/(\alpha+1)]$$

giving

$$\overline{u} = \frac{x^{\alpha+1}}{s^2} - \frac{(\alpha+1)}{s^3}\left\{1 - \exp[-sx^{\alpha+1}/(\alpha+1)]\right\}$$

Inverting the transform with the help of the shifting theorem:

$$u(x,t) = tx^{\alpha+1} - \frac{1}{2}(\alpha+1)t^2 + \frac{(\alpha t + t - x^{\alpha+1})^2}{2(\alpha+1)}H\left(t - \frac{x^{\alpha+1}}{\alpha+1}\right).$$

(2) Inserting the function into the definition of the Laplace transform and using the properties of the $\delta-$function gives the desired result. Laplace transforming the PDE:

$$\overline{u}_{xx} - s^2\overline{u} = -\frac{1}{v}e^{-sx/v},$$

Hence if $v \neq 1$,

$$\overline{u}(x,s) = \frac{v(e^{-sx} - e^{-sx/v})}{s^2(1-v^2)}.$$

Inverting the transform gives

$$u(x,t) = \frac{1}{v^2-1}[v(t-x)H(t-x) - (vt-x)H(vt-x)].$$

At a representative time, the solution for $c > 1$ forms a triangle above $0 < x < t$, with a peak at $x = vt$ and $u = 0$ for $x > t$ if $v < 1$, or above $0 < x < vt$, with a peak at $x = t$ and $u = 0$ for $x > vt$ if $v > 1$. For $v = 1$, we find $\overline{u}(x,s) = xe^{-sx}/(2s)$, and so $u(x,t) = \frac{1}{2}xH(t-x)$. The solution now is a right-angle triangle above $0 < x < t$.

(3) The characteristic equations are

$$\frac{dx}{dt} = \frac{t}{x} \quad \& \quad \frac{du}{dt} = -\frac{tu^2}{x}.$$

Hence, given that $x = x_0$ and $u = f(x_0)$ at $t = 0$,

$$x^2 = x_0^2 + t^2 \quad \& \quad \frac{1}{u} = \frac{1}{f(x_0)} + x_0\sqrt{1 + \frac{t^2}{x_0^2}} - x_0.$$

Thus

$$u = \frac{f(x\sqrt{1+t^2/x^2})}{1 + [x - x\sqrt{1+t^2/x^2}]f(x\sqrt{1+t^2/x^2})}$$

It's a bit of a mystery what happens in $-t < x < t$. Here the chacteristics are $x = \pm\sqrt{t^2 - t_0^2}$ and begin on the $t-$axis, where information is not provided. Hence, without a further assumption, we cannot provide the solution here.

(4) The characteristic equations are

$$\frac{dx}{dt} = x^{2n} \quad \& \quad \frac{du}{dt} = x^{2m}.$$

Hence, given that $x = x_0$ and $u = 0$ at $t = 0$, and that $2n - 1 \neq 0$ and $2n - 2m - 1 \neq 0$,

$$x^{-(2n-1)} = x_0^{-(2n-1)} - (2n-1)t \qquad \& \qquad u = \frac{[x^{-(2n-1)} + (2n-1)t]^{(2n-2m-1)/(2n-1)} - x^{-(2n-2m-1)}}{2n - 2m - 1}.$$

(5) The characteristic equations are

$$\frac{dx}{dt} = x^{-\alpha} \quad \& \quad \frac{du}{dt} = x^{1+\alpha}.$$

Hence, if $x = x_0$ and $u = 0$ at $t = 0$,

$$x^{\alpha+1} = x_0^{\alpha+1} + (\alpha+1)t \quad \& \quad u = x_0^{\alpha+1}t + \frac{1}{2}(\alpha+1)t^2,$$

which is the case for $x^{\alpha+1} - (\alpha+1)t > 0$. But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = 0$, we find instead that

$$x^{\alpha+1} = (\alpha+1)(t - t_0) \quad \& \quad u = \frac{1}{2}(\alpha+1)(t-t_0)^2,$$

applying for $x^{\alpha+1} - (\alpha+1)t < 0$. Eliminating either x_0 or t_0 now gives the earlier result.

Coursework 5: Laplace transform and characteristics problems 2017

(1) Establish that $\mathcal{L}\{t^n\} = s^{-n-1}n!$. Use a Laplace transform to solve

$$xu_t + u_x = x^3,$$

for $x \geq 0$, subject to $u(x,0) = 0$ and $u(0,t) = 0$.

(2) Establish that $\mathcal{L}\{\delta(x - vt)\} = v^{-1}e^{-sx/v}$ and $\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$. Use a Laplace transform to solve

$$u_{tt} = c^2[u_{xx} + \delta(x - t)], \qquad u(0,t) = u(x,0) = u_t(x,0) = 0 \ \& \ u \to 0 \text{ as } x \to \infty,$$

for $c > 1$ and $c = 1$. Plot your solutions at some representative $t > 0$.

(3) Re-solve problem (1) using the method of characteristics.

(4) Using the method of characteristics, solve

$$u_t + t^2 u_x = -u^3$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x,0) = f(x)$.

Coursework 5: Solutions to actual problems

(1) *Solution:* Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as $\mathrm{Re}(s) > 0$). Laplace transforming the PDE and boundary condition:

$$sx\overline{u} + \overline{u}_x = \frac{x^3}{s}, \quad \overline{u}(0,s) = 0.$$

Hence

$$\overline{u} = \frac{x^2}{s^2} - \frac{2}{s^3} + \frac{2}{s^3}e^{-sx^2/2}$$

(using an integrating factor of $e^{sx^2/2}$). Inverting the transform using a shifting theorem:

$$u(x,t) = t(x^2 - t) + \frac{1}{4}(2t - x^2)^2 H(2t - x^2).$$

(2) Inserting the functions into the definition of the Laplace transform gives the desired results. Laplace transforming the PDE:

$$c^2\overline{u}_{xx} - s^2\overline{u} = -c^2 e^{-sx},$$

Hence if $c \neq 1$,

$$\overline{u}(x,s) = \frac{c^2(e^{-sx} - e^{-sx/c})}{s^2(1-c^2)}.$$

Inverting the transform gives

$$u(x,t) = \frac{c}{(1-c^2)}[c(t-x)H(t-x) - (ct-x)H(ct-x)].$$

At a representative time, the solution for $c > 1$ forms a triangle above $0 < x < ct$, with $u = 0$ for $x > ct$. For $c = 1$, we find $\overline{u}(x,s) = xe^{-sx}/(2s)$, and so $u(x,t) = \frac{1}{2}xH(t-x)$. The solution now is a right-angle triangle above $0 < x < t$.

(3) The characteristic equations are

$$\frac{dx}{dt} = \frac{1}{x} \quad \& \quad \frac{du}{dt} = x^2.$$

Hence, if the characteristic intersects $x = x_0$ and $u = 0$ at $t = 0$,

$$x^2 = x_0^2 + 2t \quad \& \quad u = x_0^2 t + t^2,$$

giving $u(x,t) = tx^2 - t^2$, as found previously for $2t < x^2$. But if the characteristic leaves $x = 0$ at $t = t_0$ with $u = 0$, we find instead that

$$x^2 = 2(t - t_0) \quad \& \quad u = (t - t_0)^2 \equiv \frac{x^4}{4},$$

which is the earlier result for $2t > x^2$.

(4) The characteristic equations are

$$\frac{dx}{dt} = t^2 \quad \& \quad \frac{du}{dt} = -u^3.$$

Hence, if $x = x_0$ and $u = u_0$ at $t = 0$,

$$x = x_0 + \frac{1}{3}t^3 \quad \& \quad u^2 = \frac{u_0^2}{1 + 2u_0^2 t}.$$

But $u(x,0) = f(x)$ and so

$$u(x,t) = \frac{f(x - t^3/3)}{\sqrt{1 + 2t[f(x - t^3/3)]^2}}.$$