Coursework 5: Laplace transform and characteristics problems

(1) Establish that $L\{t^n\} = s^{-n-1}n!$. Use a Laplace transform to solve

$$xu_t + u_x = x^3,$$

for $x \geq 0$, subject to $u(x, 0) = 0$ and $u(0, t) = 0$.

(2) Establish that $L\{\delta(x - vt)\} = v^{-1}e^{-sx/v}$ and $L\{f(t - a)H(t - a)\} = e^{-as}\mathcal{F}(s)$. Use a Laplace transform to solve

$$u_{tt} = c^2[u_{xx} + \delta(x - t)], \quad u(0, t) = u(x, 0) = u_t(x, 0) = 0 \& u \to 0 \text{ as } x \to \infty,$$

for $c > 1$ and $c = 1$. Plot your solutions at some representative $t > 0$.

(3) Re-solve problem (1) using the method of characteristics.

(4) Using the method of characteristics, solve

$$u_t + t^2u_x = -u^3$$

for $-\infty < x < \infty$ and $t > 0$, subject to $u(x, 0) = f(x)$. 
Coursework 5: Sample Laplace transform and characteristics problems

(1) Establish that \( L\{e^{at}\} = (s - a)^{-1} \). Use a Laplace transform to solve
\[
    u_t + xu_x = x^2,
\]
for \( x \geq 0 \), subject to \( u(x, 0) = 0 \) and \( u(0, t) = 0 \).

**Solution:** Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as \( \text{Re}(s) > a \)). Laplace transforming the PDE and boundary condition:
\[
    s\pi + x\pi_x = \frac{x^2}{s}, \quad \pi(0, s) = 0.
\]
Hence \( \pi = x^2/[s(s + 2)] \) (using an integrating factor of \( x^a \), and then the boundary condition to discard the homogeneous solution). Inverting the transform using a partial fraction gives
\[
    u(x, t) = \frac{1}{2}x^2(1 - e^{-2t}).
\]

(2) Establish that \( L\{\cos at\} = s/(s^2 + a^2) \) and \( L\{\sin at\} = a/(s^2 + a^2) \). Use a Laplace transform to show that the solution to
\[
    u_t + cu_x = \cos \omega t \delta(x - t), \quad u(0, t) = u(x, 0) = 0 \& x > 0,
\]
for \( c > 1 \) is
\[
    u(x, t) = \frac{\cos[\Omega(x - ct)/c][H(ct - x) - H(t - x)]}{c - 1},
\]
where \( \Omega = \omega c/(c - 1) \). Show that \( u(x, t) = \omega^{-1}\sin \omega t \delta(t - x) \) for \( c = 1 \).

**Solution:** Inserting the functions into the definition of the Laplace transform and integrating by parts connects the transforms together and then gives the desired result (as long as \( \text{Re}(s) > 0 \)). Laplace transforming the PDE and boundary condition:
\[
    c\pi_x + s\pi = e^{-sx}\cos \omega x, \quad \pi(0, s) = 0.
\]
Hence
\[
    \pi(x, s) = \frac{s(e^{-sx}/c - e^{-sx}\cos \omega x) + \Omega e^{-sx}\sin \omega x}{(c - 1)(s^2 + \Omega^2)}.
\]
Inverting the transform and using a trig relation gives the first result. For \( c = 1 \), we find \( \pi(x, s) = \omega^{-1}e^{-sx}\sin \omega x \), and inverting the transform gives the second result.

(3) Re-solve problem (1) using the method of characteristics.

**Solution:** The characteristic equations are
\[
    \frac{dx}{dt} = x \quad \& \quad \frac{du}{dt} = x^2.
\]
Hence, along the characteristic curves that begin at \( x = x_0 \) at \( t = 0 \), where \( u = 0 \),
\[
    x = x_0 e^t \quad \& \quad u = \frac{1}{2}x_0^2(e^{2t} - 1),
\]
which give the same solution as in (1) on eliminating \( x_0 \).
(4) Using the method of characteristics, solve

\[ u_t + x^2 u_x = -u^2 \]

for \(-\infty < x < \infty\) and \(t > 0\), subject to \(u(x, 0) = f(x)\).

**Solution:** The characteristic equations are

\[ \frac{dx}{dt} = x^2 \quad \& \quad \frac{du}{dt} = -u^2. \]

Hence, if \(x = x_0\) and \(u = u_0\) at \(t = 0\),

\[ x = \frac{x_0}{1 - x_0 t} \quad \& \quad u = \frac{u_0}{1 + u_0 t}. \]

But \(u(x, 0) = f(x)\) and so

\[ u(x, t) = \frac{f(x/(1 + xt))}{1 + tf(x/(1 + xt))}. \]
(1) Solution: Inserting the function into the definition of the Laplace transform and integrating gives the desired result (as long as \( \text{Re}(s) > 0 \)). Laplace transforming the PDE and boundary condition:

\[
sx\overline{u} + \overline{u}_x = \frac{x^3}{s}, \quad \overline{u}(0, s) = 0.
\]

Hence

\[
\overline{u} = \frac{x^2}{s^2} - \frac{2}{s^3} + \frac{2}{s}e^{-sx^2/2}
\]

(using an integrating factor of \( e^{sx^2/2} \)). Inverting the transform using a shifting theorem:

\[
u(x, t) = t(x^2 - t) + \frac{1}{4}(2t - x^2)^2H(2t - x^2).
\]

(2) Inserting the functions into the definition of the Laplace transform gives the desired results. Laplace transforming the PDE:

\[
c^2\overline{u}_{xx} - s^2\overline{u} = -c^2e^{-sx},
\]

Hence if \( c \neq 1 \),

\[
\overline{u}(x, s) = \frac{c^2(e^{-sx} - e^{-sx/c})}{s^2(1 - c^2)}.
\]

Inverting the transform gives

\[
u(x, t) = \frac{c}{(1 - c^2)}[c(t - x)H(t - x) - (ct - x)H(ct - x)].
\]

At a representative time, the solution for \( c > 1 \) forms a triangle above \( 0 < x < ct \), with \( u = 0 \) for \( x > ct \). For \( c = 1 \), we find \( \overline{u}(x, s) = xe^{-sx}/(2s) \), and so \( u(x, t) = \frac{1}{2}xH(t - x) \). The solution now is a right-angle triangle above \( 0 < x < t \).

(3) The characteristic equations are

\[
\frac{dx}{dt} = \frac{1}{x} \quad \text{&} \quad \frac{du}{dt} = x^2.
\]

Hence, if the characteristic intersects \( x = x_0 \) and \( u = 0 \) at \( t = 0 \),

\[
x^2 = x_0^2 + 2t \quad \text{&} \quad u = x_0^2t + t^2,
\]

giving \( u(x, t) = tx^2 - t^2 \), as found previously for \( 2t < x^2 \). But if the characteristic leaves \( x = 0 \) at \( t = t_0 \) with \( u = 0 \), we find instead that

\[
x^2 = 2(t - t_0) \quad \text{&} \quad u = (t - t_0)^2 \equiv \frac{x^4}{4},
\]

which is the earlier result for \( 2t > x^2 \).

(4) The characteristic equations are

\[
\frac{dx}{dt} = t^2 \quad \text{&} \quad \frac{du}{dt} = -u^3.
\]

Hence, if \( x = x_0 \) and \( u = u_0 \) at \( t = 0 \),

\[
x = x_0 + \frac{1}{3}t^3 \quad \text{&} \quad u^2 = \frac{u_0^2}{1 + 2u_0^2t}.
\]

But \( u(x, 0) = f(x) \) and so

\[
u(x, t) = \frac{f(x - t^3/3)}{\sqrt{1 + 2t[f(x - t^3/3)]^2}}.
\]