Coursework 3

The exercises here follow the steps taken in lectures for PDEs in cylindrical or spherical geometry, with Bessel functions or Legendre polynomials as solutions. i.e. the warm-up problems were done in class...

(1) Consider the PDE

\[ u_{tt} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right), \quad 0 \leq r \leq 1 \quad 0 \leq \theta \leq \pi \]

with boundary condition, \( u(1, \theta, t) = 0 \). Use separation of variables to demonstrate that the solution of the initial-value problem, \( u(r, \theta, 0) = 0 \) and \( u_t(r, \theta, 0) = g(r, \theta) \), can be written as the superposition of normal modes,

\[ u = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} A_{nj} \sin(\omega_{nj} t) F_j(r) G_n(\cos \theta), \]

where you should determine the functions \( F_j(r) \) and \( G_n(\cos \theta) \), along with the normal-mode frequency \( \omega_{nj} \).

Provide an expression for the normal-mode amplitudes \( A_{nj} \) in terms of integrals over the initial condition \( g(r, \theta) \). Make this expression as explicit as you can by using the helpful information given below and proving that

\[ \frac{1}{2} [ J_2^2(k) + J_0^2(k) ] = \int_0^1 r J_0^2(kr) dr. \]

Using Legendre’s equation, show that

\[ \int_{-1}^{1} P_0(x) dx = 2 \quad \text{and} \quad \int_{-1}^{1} P_n(x) dx = 0 \quad \text{if} \quad n > 0. \]

Hence, if

\[ \int_0^\pi g(r, \theta) \sin \theta \, d\theta = 0, \]

argue that the sum over \( n \) starts with \( n = 1 \).

(2) The Schrödinger equation and Hermite polynomials: Consider the PDE,

\[ i\phi_t = \phi_{xx} - V(x)\phi, \]

where \( V(x) \) is a prescribed potential function and \(-\infty < x < \infty\). \( \phi(x, t) \) is known as the wavefunction. The separation of variables, \( \phi = e^{iE t} X(x) \), reduces the PDE to the eigenvalue problem,

\[ X'' + (E - V) X = 0, \]

where \( X(x) \to 0 \) for \( x \to \pm\infty \).

(a) For the potential of the simple harmonic oscillator, \( V(x) = x^2/4 \), introduce the transformation, \( X = e^{-x^2/4} y(x) \), to show that the solutions satisfy Hermite’s ODE (see below).

(b) By posing the polynomial solution, \( y(x) = \sum_{m=0}^{\infty} a_m x^m \), obtain a recurrence relation for the coefficients, \( a_m \), and hence determine for what values of \( E \) the series terminates.

(c) Calculate \( H_n(x) \) for \( n = 2, 3, \ldots, 5 \), using the recurrence relation of (b). Verify your results using the recursion relation for the polynomials themselves (given below), and Rodrigues formula.

(d) By writing Hermite’s ODE in standard Sturm-Liouville form, establish that the weight function is indeed \( \sigma = e^{-x^2/2} \); what is \( p(x) \) and how is \( E \) related to the Sturm-Liouville eigenvalue? What kind of boundary conditions are being imposed on the Sturm-Liouville problem for \( y(x) \)?
(e) Evaluate the integral
\[ \int_{-\infty}^{\infty} H_n(x) \frac{d^n}{dx^n} (e^{-x^2/2}) dx \]
by repeatedly integrating by parts. Hence verify the integral relation at the bottom of the page, given Rodrigues formula.

(f) Collect together the previous results to write down a solution for the wavefunction \( \phi(x,t) \), when the initial condition is \( \phi(x,0) = f(x) \), expressing any coefficients as integrals involving the Hermite polynomials and \( f(x) \).

(g) Find the wavefunction for all time for \( \phi(x,0) = (4x^2 + x^4)e^{-x^2/4} \) and \( \phi(x,0) = e^{-x^2/4-x} \).

**Helpful information:**

Bessel’s equation is
\[ r^2y'' + ry' + (k^2r^2 - m^2)y = 0, \]
and has the solution, \( y(r) = J_m(kr) \), which is regular at \( r = 0 \). \( J_0(z) \) and \( J_1(z) \) satisfy the relations
\[ \frac{d}{dz}J_0(z) = -J_1(z), \quad \frac{d}{dz}[zJ_1(z)] = zJ_0(z). \]

Legendre’s equation is
\[ \frac{d}{dx} \left[(1-x^2) \frac{dy}{dx}\right] + \lambda y = 0; \]
the solutions that are regular at \( x = \pm 1 \) are \( \lambda = n(n+1) \) and \( y = P_n(x) \) (the Legendre polynomial of degree \( n \)), with \( n = 0, 1, 2, ... \) Also, \( P_n(1) = 1 \) and
\[ \int_{-1}^{1} P_n^2(x)dx = \frac{2}{1+2n}. \]

A summary of the properties of Hermite polynomials:

Hermite’s ODE: \( y'' - xy' + \lambda y = 0 \)

Weight function: \( \sigma(x) = e^{-x^2/2} \)

Interval: \(-\infty < x < \infty\)

Regular solutions: \( y(x) = H_n(x) \) and \( \lambda = n \)

Normalization: The leading coefficient of the polynomial is unity. i.e. \( H_n(x) = x^n + ... \), so \( H_0(x) = 1 \) and \( H_1(x) = x \), etc.

Recurrence relation: \( H_{n+1} - xH_n + nH_{n-1} = 0 \)

Rodrigues formula:
\[ H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2/2}). \]

Integral relation:
\[ \int_{-\infty}^{\infty} [H_n(x)]^2 e^{-x^2/2} dx = n!\sqrt{2\pi}. \]