Coursework 2: Sturm-Liouville problems and Bessel functions

Hand in solutions to the questions on page 1 only; pages 2 and 3 contain supplementary “warm-up” problems to practice on for your own enjoyment.

(1). Consider the axisymmetric heat equation,

\[ u_t = \frac{1}{r} (ru_r)_r \]

in \( r \leq R \), subject to \( u(R,t) = 0 \) and \( u(r,t) \) regular at the origin. Determine the Sturm-Liouville (SL) problem satisfied by the radial part of the separable solution, \( u(r,t) = X(r)T(t) \), establishing the form of the functions \( p(r) \), \( q(r) \) and \( \sigma(r) \) in the ODE and stating the boundary conditions and how the eigenvalue is related to the separation constant of the PDE. Show that the eigenfunctions of the SL problem are Bessel functions, and write the eigenvalue in terms of the zeros of \( J_0(z) \). Given \( u(r,0) = f(r) \), express the solution to the PDE in terms of Bessel functions and their integrals.

(2). Using the method of separation of variables, solve the wave equation inside the unit disk, \( r \leq 1 \), applying the boundary condition, \( u(1,\theta,t) = 0 \), and initial conditions,

\[ u(r,\theta,0) = \frac{1}{2} f_0(r) + \sum_{m=1}^{\infty} f_m(r) \cos m\theta \quad \text{and} \quad u_t(r,\theta,0) = 0, \]

expressing your result in terms of Bessel functions and their integrals.

Bessel’s equation is

\[ x^2 y'' + xy' + (k^2 x^2 - m^2)y = 0, \]

and has the two solutions, \( y = J_m(kx) \) and \( Y_m(kx) \), of which only the former is regular at \( z = 0 \). For your enjoyment, here is a picture of \( J_m(z) \) for \( m = 0, 1 \) and 2.

Remember, Bessel functions are our friends.
Warm-up problems

(1). The equation of motion of a hanging, heavy chain is

\[ u_{tt} = \frac{\partial}{\partial x} \left( x \frac{\partial u}{\partial x} \right), \]

where \( u(x, t) \) is the horizontal deflection at height \( x \) and time \( t \) (the tension in the chain varies with height due to the overlying weight). The end at \( x = 0 \) is free, whereas the end at \( x = l \) is fixed, so that \( u_x(0, t) = u(l, t) = 0 \). Using separation of variables reduce the PDE to two equivalent ODEs. Show that the spatial dependence of the solution is given by the Bessel function, \( J_0(z) \). 

Hint: the transformation \( x = cz^2 \) may prove helpful, for some constant \( c \).

Given that the zeros of \( J_0(z) \) are \( z = z_1, z_2, \ldots, z_n, \ldots \), write down a general solution of the PDE in terms of a sum over Bessel functions with unspecified coefficients. If \( u(x, 0) = 0 \) and \( u_t(x, 0) = f(x) \), express those coefficients in terms of integrals of \( J_0(z) \).

Separation of variables: \( u = X(x)T(t) \), with

\[ xX'' + X' + \lambda_n^2 X = 0, \quad T = a_n \cos \lambda_n t + b_n \sin \lambda_n t. \]

Making the suggested change of variable and choosing \( c = 1/(4\lambda_n^2) \), leads to

\[ X_{zz} + \frac{1}{z}X_z + X = 0 \quad \longrightarrow \quad X = J_0(z) = J_0(2\lambda_n \sqrt{x}), \]

on using the regularity of \( J_0(z) \) at \( z = 0 \). The other boundary condition implies that \( \lambda_n = z_n/2\sqrt{l} \), where \( z_n \) is the \( n^{th} \) zero of \( J_0(z) \). Thus,

\[ u = \sum_{n=1}^{\infty} \left( a_n \cos \lambda_n t + b_n \sin \lambda_n t \right) J_0(2\lambda_n \sqrt{x}), \]

with the given initial condition, \( a_n = 0 \) and \( b_n \) must be computed from a suitable expansion in Bessel functions. Given that the equation for \( X(x) \) is a Sturm-Liouville problem with weight \( \sigma(x) = 1 \), the \( J_0 \)'s form an orthogonal basis set, and we arrive at

\[ b_n = \frac{2\sqrt{l}}{z_n} \int_0^l f(x) J_0(z_n \sqrt{x/l}) dx \int_0^l J_0^2(z_n \sqrt{x/l}) dx. \]

(2). Using the method of separation of variables, solve Laplace’s equation inside the cylinder, \( 0 \leq r \leq R, 0 \leq \theta \leq 2\pi, 0 \leq z \leq L \), in cylindrical polar coordinates \( (r, \theta, z) \), applying the boundary condition, \( u(R, \theta, z) = 0, u(r, \theta, 0) = 0 \) and

\[ u(r, \theta, L) = F(r, \theta) = \sum_{m=1}^{\infty} F_m(r) \sin m\theta \]

expressing your result in terms of Bessel functions (including any constants of integration).

The PDE to solve is

\[ \frac{1}{r} (ru_r)_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0. \]
We put \( u = X(r)Y(\theta)Z(z) \) and rewrite the PDE as
\[
\frac{1}{rX}(rX)_r + \frac{1}{r^2Y}Y_{\theta\theta} = -\frac{Z_{zz}}{Z}.
\]
The right-hand side is a function of \( z \) alone, whereas the left-hand side is a function of \( r \) and \( \theta \), so both must equal a separation constant, \(-k^2\). Hence
\[
\frac{r}{X}(rX)_r + r^2k^2 = -\frac{Y_{\theta\theta}}{Y}.
\]
The right-hand side is now a function of \( \theta \), the left is a function of \( r \); we put both equal the separation constant \( m^2 \). Consequently,
\[
Z_{zz} = k^2Z, \quad \text{and} \quad Y_{\theta\theta} = -m^2Y,
\]
Thus,
\[
Z = \sinh kz, \quad Y = \cos m\theta \text{ or } \sin m\theta, \quad m = 0, 1, 2, \ldots
\]
(since \( Y(\theta) \) must be \( 2\pi \)-periodic). Finally,
\[
(rX)_r - \frac{m^2}{r}X + rk^2X = 0,
\]
which, along with the boundary conditions, \( X(r) \) regular at \( r = 0 \) and \( X(R) = 0 \), determine a Sturm-Liouville problem for \( X(r) \) and eigenvalue \( k^2 \), with \( p(r) = r, \); \( q(r) = -m^2/r \) and \( \sigma(r) = r \). For each \( m \), there is an infinite number of solutions, \( k = k_{m,n} \) and \( X(r) = X_{m,n}(r), \quad n = 1, 2, \ldots \)
The general solution of the PDE is therefore
\[
u(r, \theta, z) = \sum_{n=1}^{\infty} \left[ \frac{1}{2}a_{0,n}X_{0,n}(r) \sinh(k_{0,n}z) + \sum_{m=1}^{\infty} (a_{m,n} \cos m\theta + b_{m,n} \sin m\theta)X_{m,n}(r) \sinh(k_{m,n}z) \right].
\]
Comparing the ODE of the Sturm-Liouville problem with Bessel’s equation, we see that
\[
X_{m,n}(r) \equiv J_m(k_{m,n}r) \quad \text{and} \quad k_{m,n} = \frac{z_{m,n}}{R}
\]
where \( z_{m,n} \) is the \( n^{th} \) zero of \( J_m(z) \). Given also the boundary condition at \( z = L \) (a sine series in \( \theta \)), we have \( a_{j,n} = 0 \) and
\[
u(r, \theta, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{m,n} \sin m\theta J_m(k_{m,n}r) \sinh(k_{m,n}z).\]
Finally,
\[
F_m(r) = \sum_{n=1}^{\infty} b_{m,n} J_m(k_{m,n}r) \sinh(k_{m,n}L),
\]
and so
\[
b_{m,n} = \frac{\int_{0}^{R} F_m(r)J_m(k_{m,n}r)rdr}{\int_{0}^{R} [J_m(k_{m,n}r)]^2rdr}.
\]
Solutions to the real problems

(1). Separate variables: \( u = X(r)T(t) \), giving

\[
\frac{1}{rX}(rX_r)_r = \frac{T_t}{T} = -k^2,
\]

where \(-k^2\) is the separation constant. Hence

\[(rX)_r + k^2 rX = 0 \quad \text{and} \quad T = e^{-k^2t}.\]

The first equation is the ODE of a Sturm-Liouville (SL) problem with \( p(r) = \sigma(r) = r \), \( q(r) = 0 \)
and eigenvalue \( k^2 \). Comparison with Bessel’s equation and imposition of \( X(R) = 0 \) indicates that

\[X(r) = J_0(kr) \quad \text{and} \quad J_0(kR) = 0.\]

Denoting \( z_n \) as the \( n \)th zero of \( J_0(z) \), \( n = 1, 2, ... \), we find the SL eigenvalues, \( k_n = z_n/R \), and
eigenfunctions, \( X_n(r) = J_0(k_n r) \). Hence,

\[u(r, t) = \sum_{n=1}^{\infty} c_n e^{-k_n^2 t} J_0(k_n r).\]

Finally, we apply the initial condition:

\[f(r) = \sum_{n=1}^{\infty} c_n J_0(k_n r) \quad \longrightarrow \quad c_n = \frac{\int_0^R f(r) J_0(k_n r) r dr}{\int_0^R [J_0(k_n r)]^2 r dr}.\]

(2). The PDE to solve is

\[u_{tt} = \frac{1}{r}(ru_r)_r + \frac{1}{r^2} u_{\theta\theta}.\]

We put \( u = X(r)Y(\theta)T(t) \) and rewrite the PDE as

\[
\frac{1}{rX}(rX_r)_r + \frac{1}{r^2 Y} Y_{\theta\theta} = \frac{T_{tt}}{T}.
\]

The right-hand side is a function of \( t \) alone, whereas the left-hand side is a function of \( r \) and \( \theta \), so
both equal a separation constant, \(-k^2\). Hence

\[
\frac{r}{X}(rX_r)_r + r^2 k^2 = -\frac{Y_{\theta\theta}}{Y}.
\]

The right-hand side is now a function of \( \theta \), the left is a function of \( r \); we put both equal the
separation constant \( m^2 \). Consequently,

\[T_{tt} = -k^2 T, \quad \text{and} \quad Y_{\theta\theta} = -m^2 Y,\]

Thus,

\[T = \cos kt, \quad Y = \cos m\theta \text{ or } \sin m\theta, \quad m = 0, 1, 2, ...\]
since $u_t(r, \theta, 0) = 0$ (or $T_t(0) = 0$) and $Y(\theta)$ must be $2\pi$-periodic. Finally,

$$(rX_r)_r - \frac{m^2}{r}X + rk^2X = 0,$$

which, along with the boundary conditions, $X(r)$ regular at $r = 0$ and $X(1) = 0$, determine a Sturm-Liouville problem for $X(r)$ and eigenvalue $k^2$, with $p(r) = r$, $q(r) = -m^2/r$ and $\sigma(r) = r$. For each $m$, there is an infinite number of solutions, $k = k_{m,n}$ and $X(r) = X_{m,n}(r)$, $n = 1, 2, \ldots$

The general solution of the PDE is therefore

$$u(r, \theta, z) = \sum_{n=1}^{\infty} \left[ \frac{1}{2}a_{0,n}X_{0,n}(r) \cos(k_{0,n}t) + \sum_{m=1}^{\infty} (a_{m,n} \cos m\theta + b_{m,n} \sin m\theta)X_{m,n}(r) \cos(k_{m,n}t) \right].$$

Comparing the ODE of the Sturm-Liouville problem with Bessel’s equation, we see that

$$X_{m,n}(r) \equiv J_m(k_{m,n}r) \quad \text{and} \quad k_{m,n} = z_{m,n}$$

where $z_{m,n}$ is the $n^{th}$ zero of $J_m(z)$.

Finally, we observe that $u(r, \theta, 0)$ is a cosine series in $\theta$, so $b_{m,n} = 0$, and

$$u(r, \theta, z) = \sum_{n=1}^{\infty} \left[ \frac{1}{2}a_{0,n}J_0(k_{0,n}r) \cos(k_{0,n}t) + \sum_{m=1}^{\infty} a_{m,n} \cos m\theta J_m(k_{m,n}r) \cos(k_{m,n}t) \right].$$

Finally, demanding $u(r, \theta, 0) = f(r, \theta) = \frac{1}{2}f_0(r) + \sum_m f_m(r) \cos m\theta$, implies

$$f_m(r) = \sum_{n=1}^{\infty} a_{m,n}J_m(k_{m,n}r)$$

(for $m = 0, 1, 2, \ldots$), and so

$$a_{m,n} = \frac{\int_0^R f_m(r)J_m(k_{m,n}r)rdr}{\int_0^R [J_m(k_{m,n}r)]^2rdr}.$$