Coursework 1: Separation of Variables

Hand in solutions to the questions on pages 1-2 only; the later pages contain supplementary "warm-up" problems to practice on for your own enjoyment.

1. Solve the heat equation,

$$u_t = u_{xx},$$

for $0 \le x \le \frac{1}{2}\pi$ and $t \ge 0$, subject to the boundary conditions,

$$u_x(0,t) = u(\tfrac{1}{2}\pi, t) = 0,$$

and the intial condition,

$$u(x,0) = 1.$$

Briefly justify your solution using Fourier series theory. The MATLAB code pd23a.m provides a numerical solution to this problem using an in-built solver function PDEPE (output in figure 1; pd23a.png). Compare the numerical solution with your series solution, truncated to include only the first six terms, at the times and positions plotted by pd23a.m. Comment on the main discrepancy between the numerical solution and your truncated analytical solution. In the third panel, the red dashed line shows the function $\frac{4}{\pi}e^{-t}$; give a reason why this function should be close to $u(0,t)$ and $2u(\frac{1}{3}\pi, t)$.

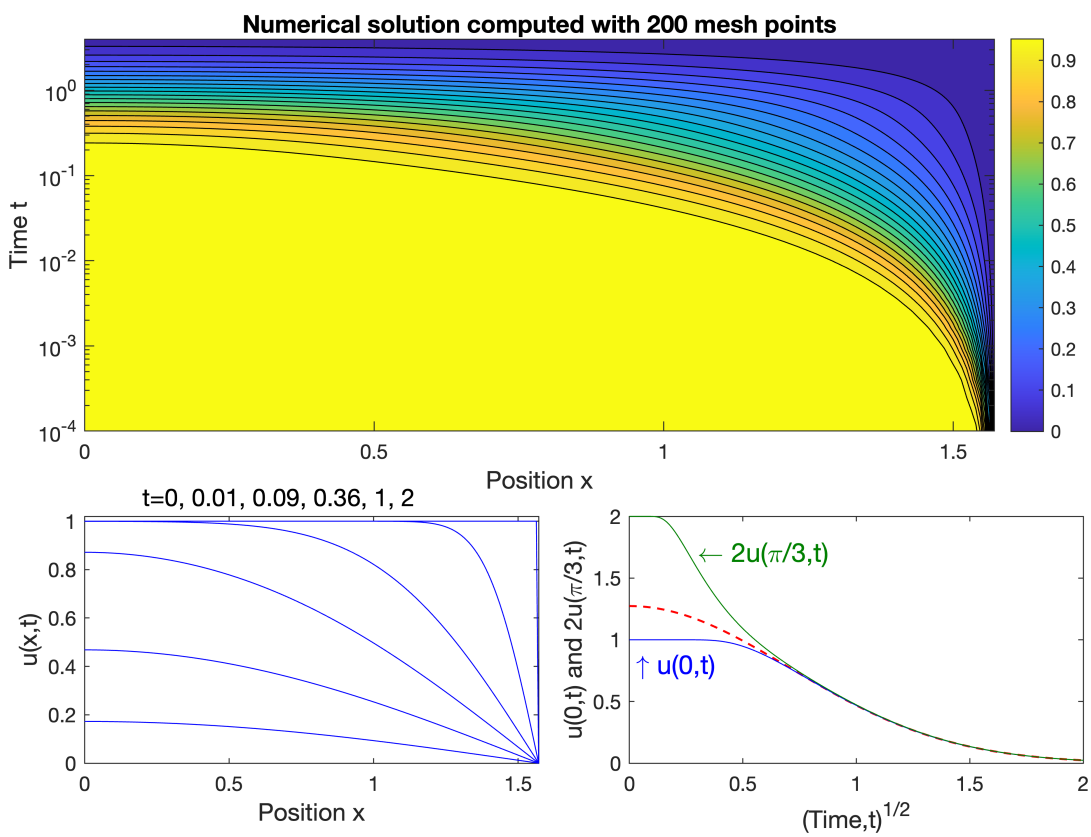

Figure 1: Output of pd23a.m; to show more of the early-time behaviour, the top panel plots time logarithmically, whereas the bottom right plot uses $\sqrt{t}$.

2. Solve the PDE

$$16r^{1/2}\frac{\partial}{\partial r}\left(r^{3/2}\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial\theta^2}+\phi=0,$$

for $1<r<R$ and $0\le\theta\le\pi$, subject to the boundary conditions that $\phi(r,0)=\phi(r,\pi)=0$ and

$$(i)\quad \phi(1,\theta)=0,\quad \phi(R,\theta)=g(\theta),$$

$$(ii)\quad \phi(1,\theta)=f(\theta),\quad \phi(R,\theta)=0$$

and

$$(iii)\quad \phi(1,\theta)=f(\theta),\quad \phi(R,\theta)=g(\theta),$$

where $f(\theta)$ and $g(\theta)$ are prescribed functions with Fourier series representations,

$$f(\theta)=\sum_{n=1}^{\infty}f_n\sin n\theta\qquad\text{and}\qquad g(\theta)=\sum_{n=1}^{\infty}g_n\sin n\theta.$$

Finally, re-solve the PDE for $R\to\infty$, with $\phi(1,\theta)=f(\theta)$ prescribed and ϕ bounded for $r\to\infty$, expressing your answer in terms of an integral involving $f(\theta)$ without any series.

3. Consider the PDE:

$$u_{ttt}=u_{xx}+q(x,t).$$

The final term $q(x,t)$ is a prescribed (*i.e.* known) source. The boundary and initial conditions are

$$u(0,t)=u(\pi,t)=u_t(x,0)=u_{tt}(x,0)=0,\qquad u(x,0)=f(x).$$

Expand u, f and q in terms of Fourier sine series (with time-dependent coefficients in the cases of $u(x,t)$ and $q(x,t)$), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

$$(i)\quad q=0,\quad f=\cos 2x.$$

$$(ii)\quad q=\cos 2x,\quad f=0,$$

$$(iii)\quad q=t\ \cos 2x,\quad f=0,$$

and

$$(iv)\quad q=t\ \cos 2x,\quad f=\cos 2x.$$

Warm-up problems

1. Solve

$$u_t = \alpha u + u_{xx}, \qquad 0 \le x \le \pi, \qquad u_x(0,t) = u_x(\pi,t) = 0,$$

where α is a constant parameter, subject to (a) $u(x,0) = \cos x$, and (b) $u(x,0) = f(x)$ for some prescribed function $f(x)$.

Using separation of variables, we have $u(x,t) = X(x)T(t)$, with

$$X_{xx} = -\lambda X, \qquad T_t = (\alpha - \lambda)T.$$

Applying $X_x(0) = X_x(\pi) = 0$ implies

$$\lambda = n^2, \qquad X = \cos nx, \qquad n = 0, 1, 2, ...$$

Hence,

$$u(x,t) = \frac{1}{2}a_0 e^{\alpha t} + \sum_{n=1}^{\infty} a_n e^{(\alpha - n^2)t} \cos nx.$$

In (a), $a_n = 0$ for $n \neq 1$ and $a_1 = 1$, so $u = e^{(\alpha-1)t}\cos x$. In (b) we extend $f(x)$ as an even function to $-\pi \le x \le \pi$, allowing us to write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad a_j = \frac{2}{\pi}\int_0^{\pi} f(x)\cos jx\, dx.$$

2. Solve $\nabla^2 u = 0$ outside the unit disk, $r \ge 1$, subject to $u(1,\theta) = f(\theta)$ and u bounded as $r \to \infty$.

The solution by separation of variables is

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} r^{-m}[a_m \cos m\theta + b_m \sin m\theta],$$

after discarding the solutions r^m and $\ln r$ which diverge for $r \to \infty$. The Fourier series expression of $f(\theta)$ furnishes the coefficients a_j and b_j as the usual integrals:

$$a_j = \frac{1}{\pi}\int_0^{2\pi} f(\hat\theta)\cos j\hat\theta\, d\hat\theta, \qquad b_j = \frac{1}{\pi}\int_0^{2\pi} f(\hat\theta)\sin j\hat\theta\, d\hat\theta.$$

Introducing these integrals into the solution, using $\cos(A-B) = \cos A\cos B + \sin A \sin B$ and $\cos z = (e^{iz} + e^{-iz})/2$, interchanging the order of the integral and the sum, and then summing the series using

$$\frac{x}{(1-x)} = \sum_{m=1}^{\infty} x^m,$$

gives, with a little algebra,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-1)f(\hat\theta)d\hat\theta}{r^2 + 1 - 2r\cos(\hat\theta - \theta)}.$$

3. Solve

$$u_t = \alpha u - u_{xxxx}, \qquad 0 \le x \le \pi \qquad u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \qquad u(x,0) = f(x),$$

and then

$$u_t = \alpha u - u_{xxxx} + g(x), \qquad 0 \le x \le \pi \qquad u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \qquad u(x,0) = f(x),$$

where α is a constant, by expanding $u(x,t)$ as a Fourier sine series with time-dependent coefficients, $B_n(t)$.

For the homogeneous problem, we set

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx.$$

From the PDE,

$$\sum_{n=1}^{\infty} (\dot{B}_n - \alpha B_n + n^4 B_n) \sin nx = 0.$$

Hence, the coefficients satisfy the ODEs

$$\dot{B}_n = \alpha B_n - n^4 B_n.$$

The initial condition can be extended to $-\pi \le x \le \pi$ as an odd function, and $f(x)$ therefore written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Hence,

$$B_n(t) = b_n e^{(\alpha - n^4)t}.$$

In the inhomogeneous case, we extend $g(x)$ to $-\pi \le x \le \pi$ as an odd function:

$$g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

The ODEs extracted from the PDE are now:

$$\dot{B}_n = \alpha B_n - n^4 B_n + g_n.$$

The solutions, subject to $B_n(0) = b_n$ are

$$B_n(t) = (b_n - G_n) e^{(\alpha - n^4)t} + G_n, \qquad G_n = \frac{g_n}{n^4 - \alpha},$$

provided $\alpha^{1/4}$ is not an integer. If $\alpha = N^4$, for some integer N, then the solution above remains valid for $n \ne N$; for $n = N$, the ODE is

$$\dot{B}_N = g_N \qquad \longrightarrow \qquad B_N = b_N + g_N t.$$

More problems

1. Solve the heat equation,

$$u_t = u_{xx},$$

for $0 \le x \le \pi$ and $t \ge 0$, subject to the boundary conditions,

$$u_x(0,t) = u_x(\pi,t) = 0,$$

and the intial condition,

$$u(x,0) = e^{-x}.$$

The solution takes the form of a Fourier cosine series:

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx$$

with (using integration by parts)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} e^{-x} dx = \frac{2}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-x} \cos nx \; dx = \frac{2}{\pi}\frac{[1-(-1)^n e^{-\pi}]}{n^2+1}.$$

2. Solve Laplace's equation,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0,$$

inside the annulus, $a \le r \le b$, subject to the boundary conditions,

$$(i) \quad \phi(a,\theta) = 0, \quad \phi(b,\theta) = f(\theta),$$

$$(ii) \quad \phi(a,\theta) = g(\theta), \quad \phi(b,\theta) = 0$$

and

$$(iii) \quad \phi(a,\theta) = g(\theta), \quad \phi(b,\theta) = f(\theta),$$

where $f(\theta)$ and $g(\theta)$ are two periodic functions with Fourier series representations,

$$f(\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \qquad \text{and} \qquad g(\theta) = \frac{1}{2}g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta.$$

The solutions are

$$(i) \quad \phi(r,\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \frac{(r^n - a^{2n}r^{-n})}{(b^n - a^{2n}b^{-n})}$$

$$(ii) \quad \phi(r,\theta) = \tfrac{1}{2}g_0 \frac{\ln(r/b)}{\ln(a/b)} + \sum_{n=1}^{\infty} g_n \cos n\theta \frac{(r^n - b^{2n}r^{-n})}{(a^n - b^{2n}a^{-n})}$$

The solution to (iii) is just the sum of the two solutions above.

3. Consider the heat equation with a source term, $q(x,t)$:

$$u_t = u_{xx} + q.$$

The boundary and initial conditions are

$$u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x).$$

Expand u, f and q in terms of Fourier sine series (with time-dependent coefficients in the cases of $u(x,t)$ and $q(x,t)$), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

$$(i) \quad q = x(L-x), \quad f = 0,$$

$$(ii) \quad q = x(L-x)e^{-t}, \quad f = 0$$

and

$$(iii) \quad q = x(L-x)e^{-t}, \quad f = \sin(2\pi x/L).$$

First,

$$x(L-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with (using integration by parts)

$$b_n = \frac{4L^2}{n^3\pi^3}[1-(-1)^n].$$

Now set

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right) \qquad \& \qquad q(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

allowing the PDE to be broken down into the system of ODEs,

$$\dot{B}_n + \frac{n^2\pi^2}{L^2} B_n = Q_n,$$

subject to the initial conditions $B_n(0) = 0$ in parts (i) and (ii), and $B_2(0) = 1$ and all other $B_n(0) = 0$ in part (iii). The solutions are

$$(i) \qquad B_n = \frac{L^2 b_n}{n^2\pi^2}(1 - e^{-n^2\pi^2 t/L^2}) \qquad (Q_n = b_n)$$

$$(ii) \qquad B_n = \frac{L^2 b_n}{n^2\pi^2 - 1}(e^{-t} - e^{-n^2\pi^2 t/L^2}) \qquad (Q_n = b_n e^{-t}),$$

with $B_n(t) = t b_n e^{-t}$ if $n\pi = L$, and

$$(iii) \qquad B_n = B_n^{(ii)} \text{ if } n \neq 2 \quad \& \quad B_2 = B_2^{(ii)} + e^{-4\pi^2 t/L^2},$$

where $B_n^{(ii)}$ denotes the solution to part (ii).

Actual solutions

1. *(11 points)* After separation of variables, the solution takes the form of a Fourier cosine series:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2 t} \cos(2n-1)x$$

with

$$a_n = \frac{4}{\pi} \int_0^{\frac{1}{2}\pi} \cos(2n-1)x \, dx = \frac{4(-1)^{n+1}}{\pi(2n-1)}$$

(4 points for the solution and coefficients).

This is justified by Fourier series theory by extending $u(x,t)$ and $f(x)$ into $\frac{1}{2}\pi < x < \pi$ by $f(x) = -f(\pi - x)$ and $u(x,t) = -u(\pi - x, t)$, and then as even $2\pi-$periodic functions. This implies they can be represented as a Fourier cosine series with a sum over only the odd integers (the integrals giving the other coefficients all vanishing) (2 points for the justification)

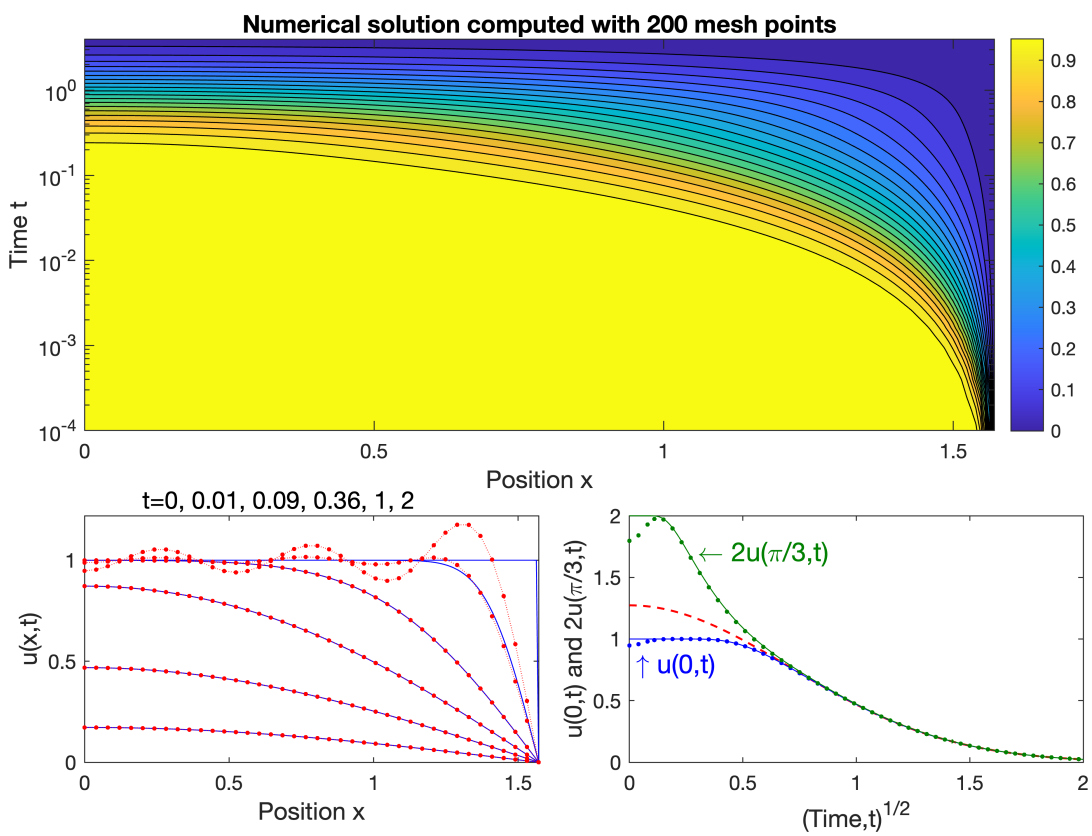

Figure 2: Output of pde22ax.m; dots are from the truncated series.

pd23ax.m and pd23ax.png compare the truncated sum with the numerical solution. (The code also numerically computes the integrals for a_n, and compares this with the analytical result above.) The main discrepancies between the truncated analytical solution and the numerical one arise for $t = 0$, where the Fourier series is attempting to represent a function with a discontinuous derivative at $x = \frac{1}{2}\pi$. For small times, the discontinuous derivative is smoothed out, but the truncation of the series remains inaccurate, leading to a pronounced Gibbs pehonemon. The reason that $2u(\frac{1}{3}\pi, t)$

and $u(0,t)$ are close to the particular exponential function for later times is that the first term of the Fourier series solution $a_1 e^{-t}\cos x$ dominates the other terms for $t>1$ and $a_1=\frac{4}{\pi}$. (5 points; 3 for the pictures, 1 for rationalizing the discrepancy and 1 for explaining the decay rate)

2. *(10 points)* Separating variables, $\phi(r,\theta)=R(r)\Theta(\theta)$, gives

$$\Theta''+n^2\Theta=0 \qquad \& \qquad 16r^{1/2}(r^{3/2}R')'+(1-n^2)R=0$$

for some separation constant n^2. Hence $\Theta\propto\sin n\theta$ and $n=1,2,...$; the $R-$equation takes the Euler form, with solutions $R\propto r^{\pm n/4-1/4}$ (2 points). Hence,

$$(i)\quad \phi(r,\theta)=\frac{R^{1/4}}{r^{1/4}}\sum_{n=1}^{\infty}g_n\sin n\theta\frac{(r^{n/4}-r^{-n/4})}{(R^{n/4}-R^{-n/4})} \qquad \text{(2 points)}$$

$$(ii)\quad \phi(r,\theta)=\frac{1}{r^{1/4}}\sum_{n=1}^{\infty}f_n\sin n\theta\frac{(R^{n/2}r^{-n/4}-r^{n/4})}{(R^{n/2}-1)}, \qquad \text{(2 points)}$$

The solution to (iii) is just the sum of the two solutions above (1 point).

For the last part, we have

$$\phi=\sum_{n=1}^{\infty}r^{-(n+1)/4}f_n\sin n\theta.$$

Using the same manipulations as for warm-up problem 2, we see that

$$\phi(r,\theta)=\frac{1}{\pi r^{1/4}}\int_0^{\pi}\left[\frac{r^{1/4}\cos(\theta-\hat\theta)-1}{r^{1/2}+1-2r^{1/4}\cos(\theta-\hat\theta)}-\frac{r^{1/4}\cos(\theta+\hat\theta)-1}{r^{1/2}+1-2r^{1/4}\cos(\theta+\hat\theta)}\right]f(\hat\theta)d\hat\theta.$$

(3 points).

Cautionary note: In the final part, one needs to dismiss a term $B_1\sin\theta$, which is not excluded by the far-field boundary condition (that ϕ is bounded). In other words, with only a boundedness condition, there is no unique solution. One actually needs to impose $\phi\to 0$ for $r\to\infty$ to eliminate this term and reach the unique solution quoted above.

3. *(8 points)* First,

$$\cos 2x=\sum_{n=1}^{\infty}b_n\sin nx$$

with $b_n=0$ for n even and

$$b_n=\frac{2}{\pi}\int_0^{\pi}\cos 2x\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}[\sin(n+2)x+\sin(n-2)x]\,dx=\frac{2n[1-(-1)^n]}{\pi(n^2-4)} \qquad \text{(2 points)}$$

for n odd.

Now set

$$u(x,t)=\sum_{n=1}^{\infty}B_n(t)\sin nx \qquad \& \qquad q(x,t)=\sum_{n=1}^{\infty}Q_n(t)\sin nx,$$

allowing the PDE to be broken down into the system of ODEs,

$$\dddot{B}_n+n^2B_n=Q_n,$$

subject to the initial conditions $\dot B_n(0)=\ddot B_n(0)=0$, and $B_n(0)=b_n$ in parts (i) and (iv), then $B_n(0)=0$ in parts $(ii)-(iii)$ (2 points).

The solutions are

$$(i) \qquad B_n = b_n \left[\frac{1}{3} e^{-n^{2/3}t} + \frac{2}{3} e^{n^{2/3}t/2} \cos\left(\frac{\sqrt{3}}{2} n^{2/3} t \right) \right],$$

$$(ii) \qquad B_n = \frac{b_n}{n^2} \left[1 - \frac{1}{3} e^{-n^{2/3}t} - \frac{2}{3} e^{n^{2/3}t/2} \cos\left(\frac{\sqrt{3}}{2} n^{2/3} t \right) \right] \qquad (Q_n = b_n),$$

$$(iii) \qquad B_n = \frac{tb_n}{n^2} + \frac{b_n}{3n^{8/3}} \left[e^{-n^{2/3}t} - \sqrt{3} e^{n^{2/3}t/2} \sin\left(\frac{\sqrt{3}}{2} n^{2/3} t \right) - e^{n^{2/3}t/2} \cos\left(\frac{\sqrt{3}}{2} n^{2/3} t \right) \right] \qquad (Q_n = tb_n),$$

and

$$(iv) \qquad B_n = B_n^{(i)} + B_n^{(iii)}$$

where $B_n^{(i)}$ and $B_n^{(iii)}$ denote the solutions to parts (i) and (iii) (4 points). Note: the purpose of the question is not to do page-loads of algebra, so it is acceptable if briefer solutions are given using matrix inverses of coefficient matrices (to solve the linear systems arising from enforcing the initial conditions).

Cautionary note: this problem is problematic! The Fourier series cannot converge!