Coursework 1: Separation of Variables

Hand in solutions to the questions on page 1 only; the later pages contain supplementary “warm-up” problems to practice on for your own enjoyment.

1. Solve the heat equation,

\[ u_t = u_{xx}, \]

for \( 0 \leq x \leq \pi \) and \( t \geq 0 \), subject to the boundary conditions,

\[ u(0, t) = u_x(\pi, t) = 0, \]

and the intial condition,

\[ u(x, 0) = \sin x. \]

Briefly justify your solution using Fourier series theory. The MATLAB code pde20a.m provides a numerical solution to this problem using an in-built solver function PDEPE (output in figure 1; pde20a.png). Compare the numerical solution with your series solution, truncated to include only the first five terms, at the times and positions plotted by pde20a.m. Comment on the main discrepancy between the numerical solution and your truncated analytical solution.

![Figure 1: Output of pde20a.m](image)

2. Solve the PDE

\[ \frac{\partial}{\partial r} \left( r^5 \frac{\partial \phi}{\partial r} \right) + 4r^3 \frac{\partial^2 \phi}{\partial \theta^2} + 4r^3 \phi = 0, \]

for \( 1 < r < R \) and \( 0 \leq \theta \leq \pi \), subject to the boundary conditions that \( u(r, 0) = u(r, \pi) = 0 \) and

(i) \( \phi(1, \theta) = 0, \quad \phi(R, \theta) = g(\theta), \)

(ii) \( \phi(1, \theta) = f(\theta), \quad \phi(R, \theta) = 0 \)

and

(iii) \( \phi(1, \theta) = f(\theta), \quad \phi(R, \theta) = g(\theta), \)
where \( f(\theta) \) and \( g(\theta) \) are prescribed functions with Fourier series representations,

\[
f(\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \quad \text{and} \quad g(\theta) = \sum_{n=1}^{\infty} g_n \sin n\theta.
\]

Finally, re-solve the PDE for \( R \to \infty \), with \( \phi(1, \theta) = f(\theta) \) prescribed and \( r^2 \phi \) bounded for \( r \to \infty \), expressing your answer in terms of an integral involving \( f(\theta) \) without any series.

3. Consider the heat equation with a source term, \( q(x, t) \):

\[
u_t = u_{xx} + u + q.
\]

The boundary and initial conditions are

\[
u(0, t) = \nu(\pi, t) = 0, \quad \nu(x, 0) = f(x).
\]

Expand \( u, f \) and \( q \) in terms of Fourier sine series (with time-dependent coefficients in the cases of \( u(x,t) \) and \( q(x,t) \)), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

\[
(i) \quad q = \sin^2 x, \quad f = 0, \\
(ii) \quad q = \sin t \sin^2 x, \quad f = 0
\]

and

\[
(iii) \quad q = \sin t \sin^2 x, \quad f = \sin^2 x.
\]
Actual solutions

1. (6 marks) After separation of variables, the solution takes the form of a Fourier sine series:

\[ u(x, t) = \sum_{n=1}^{\infty} b_n e^{-\frac{(n-\frac{1}{2})^2}{2}t} \sin(n - \frac{1}{2})x \]

with

\[ a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \sin(n - \frac{1}{2})x \, dx = \frac{(-1)^{n+1}}{\pi} \left[ \frac{1}{n - \frac{3}{2}} - \frac{1}{n + \frac{1}{2}} \right]. \]

This is justified by Fourier series theory by extending \( u(x, t) \) and \( f(x) \) into \( \pi < x < 2\pi \) by \( f(x) = f(2\pi - x) \) and \( u(x, t) = u(2\pi - x, t) \), and then as odd periodic functions with period \( 4\pi \). This implies they can be represented as a Fourier sine series with a sum over only the odd integers.

\[ \text{pde20ax.m and pde20ax.png compare the truncated sum with the numerical solution. (The code also numerically computes the integrals for } a_n, \text{ and compares this with the analytical result above.)} \]

The main discrepancies between the truncated analytical solution and the numerical one arise for \( t = 0 \), where the Fourier series is attempting to represent a function with a discontinuous derivative at \( x = \pi \). For small times, the discontinuous derivative is smoothed out, but the truncation of the series remains inaccurate.

![Numerical solution computed with 200 mesh points](image)

**Figure 2: Output of pde20ax.m**

2. (6 marks)

\[ (i) \quad u(r, \theta) = \frac{R^2}{r^2} \sum_{n=1}^{\infty} g_n \sin n\theta \frac{(r^{2n} - r^{-2n})}{(R^{2n} - R^{-2n})} \]

\[ (ii) \quad u(r, \theta) = \frac{1}{r^2} \sum_{n=1}^{\infty} f_n \sin n\theta \frac{(R^{4n} r^{-2n} - r^{2n})}{(R^{4n} - 1)} \]

The solution to \( (iii) \) is just the sum of the two solutions above. For the last part, we have

\[ \phi = \sum_{n=1}^{\infty} r^{-2n-2} f_n \sin n\theta. \]
Using the same manipulations as for warm-up problem 2, we see that

\[ u(r, \theta) = \frac{1}{\pi r^2} \int_0^\pi \left[ \frac{r^2 \cos(\theta - \hat{\theta}) - 1}{r^4 + 1 - 2r^2 \cos(\theta - \hat{\theta})} - \frac{r^2 \cos(\theta + \hat{\theta}) - 1}{r^4 + 1 - 2r^2 \cos(\theta + \hat{\theta})}\right] f(\hat{\theta}) d\hat{\theta}. \]

3. (6 marks) First,

\[ \sin^2 x \equiv \frac{1}{2} (1 - \cos 2x) = \sum_{n=1}^\infty b_n \sin nx \]

with

\[ b_n = \frac{1}{2\pi} [1 - (-1)^n] \left[ \frac{2}{n} - \frac{1}{n+2} - \frac{1}{n-2} \right]. \]

Now set

\[ u(x, t) = \sum_{n=1}^\infty B_n(t) \sin nx \quad \& \quad q(x, t) = \sum_{n=1}^\infty Q_n(t) \sin nx, \]

allowing the PDE to be broken down into the system of ODEs,

\[ \dot{B}_n + n^2 B_n = Q_n, \]

subject to the initial conditions \( B_n(0) = 0 \) in parts (i) and (ii), and \( B_n(0) = b_n \) in part (iii). The solutions are

(i) \[ B_1 = b_1 t, \quad B_n = \frac{b_n}{n^2 - 1} (1 - e^{-(n^2 - 1)t}) \quad (Q_n = b_n) \]

(ii) \[ B_n = \frac{b_n}{1 + (n^2 - 1)^2} \left[ e^{-n^2 t} - \cos t + (n^2 - 1) \sin t \right] \quad (Q_n = b_n \sin t), \]

and

(iii) \[ B_n = B_n^{(ii)} + b_n e^{-(n^2 - 1)t}, \]

where \( B_n^{(ii)} \) denotes the solution to part (ii).
Warm-up problems

1. Solve
   \[ u_t = \alpha u + u_{xx}, \quad 0 \leq x \leq \pi, \quad u_x(0,t) = u_x(\pi,t) = 0, \]
   where \( \alpha \) is a constant parameter, subject to (a) \( u(x,0) = \cos x \), and (b) \( u(x,0) = f(x) \) for some prescribed function \( f(x) \).

   Using separation of variables, we have \( u(x,t) = X(x)T(t) \), with
   \[ X_{xx} = -\lambda X, \quad T_t = (\alpha - \lambda)T. \]
   Applying \( X_x(0) = X_x(\pi) = 0 \) implies \( \lambda = n^2 \), \( X = \cos nx \), \( n = 0, 1, 2, \ldots \)
Hence,
   \[ u(x,t) = \frac{1}{2} a_0 e^{\alpha t} + \sum_{n=1}^{\infty} a_n e^{(\alpha - n^2)t} \cos nx. \]

In (a), \( a_n = 0 \) for \( n \neq 1 \) and \( a_1 = 1 \), so \( u = e^{(\alpha - 1)t} \cos x \). In (b) we extend \( f(x) \) as an even function to \( -\pi \leq x \leq \pi \), allowing us to write
   \[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_j = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos jx \, dx. \]

2. Solve \( \nabla^2 u = 0 \) outside the unit disk, \( r \geq 1 \), subject to \( u(1,\theta) = f(\theta) \) and \( u \) bounded as \( r \to \infty \).

   The solution by separation of variables is
   \[ u(r,\theta) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} r^{-m} [a_m \cos m\theta + b_m \sin m\theta], \]
after discarding the solutions \( r^m \) and \( \ln r \) which diverge for \( r \to \infty \). The Fourier series expression of \( f(\theta) \) furnishes the coefficients \( a_j \) and \( b_j \) as the usual integrals:
   \[ a_j = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \cos j\theta \, d\theta, \quad b_j = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \sin j\theta \, d\theta. \]

Introducing these integrals into the solution, using \( \cos(A - B) = \cos A \cos B + \sin A \sin B \) and \( \cos z = (e^{iz} + e^{-iz})/2 \), interchanging the order of the integral and the sum, and then summing the series using
   \[ \frac{x}{(1 - x)} = \sum_{m=1}^{\infty} x^m, \]
gives, with a little algebra,
   \[ u(r,\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(r^2 - 1)f(\theta)d\theta}{r^2 + 1 - 2r \cos(\theta - \theta)}. \]
3. Solve

\[ u_t = \alpha u - u_{xxxx}, \quad 0 \leq x \leq \pi \quad u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad u(x, 0) = f(x), \]

and then

\[ u_t = \alpha u - u_{xxxx} + g(x), \quad 0 \leq x \leq \pi \quad u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad u(x, 0) = f(x), \]

where \( \alpha \) is a constant, by expanding \( u(x, t) \) as a Fourier sine series with time-dependent coefficients, \( B_n(t) \).

For the homogeneous problem, we set

\[ u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin nx. \]

From the PDE,

\[ \sum_{n=1}^{\infty} (\dot{B}_n - \alpha B_n + n^4 B_n) \sin nx = 0. \]

Hence, the coefficients satisfy the ODEs

\[ \dot{B}_n = \alpha B_n - n^4 B_n. \]

The initial condition can be extended to \(-\pi \leq x \leq \pi\) as an odd function, and \( f(x) \) therefore written as

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{2\pi} f(x) \sin nx \, dx. \]

Hence,

\[ B_n(t) = b_n e^{(\alpha - n^4)t}. \]

In the inhomogeneous case, we extend \( g(x) \) to \(-\pi \leq x \leq \pi\) as an odd function:

\[ g(x) = \sum_{n=1}^{\infty} g_n \sin nx. \]

The ODEs extracted from the PDE are now

\[ \dot{B}_n = \alpha B_n - n^4 B_n + g_n. \]

The solutions, subject to \( B_n(0) = b_n \) are

\[ B_n(t) = (b_n - G_n)e^{(\alpha - n^4)t} + G_n, \quad G_n = \frac{g_n}{n^4 - \alpha}, \]

provided \( \alpha^{1/4} \) is not an integer. If \( \alpha = N^4 \), for some integer \( N \), then the solution above remains valid for \( n \neq N \); for \( n = N \), the ODE is

\[ \dot{B}_N = g_N \quad \rightarrow \quad B_N = b_N + g_N t. \]
More problems

1. Solve the heat equation,
\[ u_t = u_{xx}, \]
for \(0 \leq x \leq \pi\) and \(t \geq 0\), subject to the boundary conditions,
\[ u_x(0, t) = u_x(\pi, t) = 0, \]
and the initial condition,
\[ u(x, 0) = e^{-x}. \]

The solution takes the form of a Fourier cosine series:
\[ u(x, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx \]
with (using integration by parts)
\[ a_0 = \frac{2}{\pi} \int_{0}^{\pi} e^{-x} \, dx = \frac{2}{\pi} (1 - e^{-\pi}) \]
\[ a_n = \frac{2}{\pi} \int_{0}^{\pi} e^{-x} \cos nx \, dx = \frac{2}{\pi} \frac{[1 - (-1)^n e^{-\pi}]}{n^2 + 1}. \]

2. Solve Laplace’s equation,
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0, \]
inside the annulus, \(a \leq r \leq b\), subject to the boundary conditions,
(i) \( \phi(a, \theta) = 0, \quad \phi(b, \theta) = f(\theta) \),
(ii) \( \phi(a, \theta) = g(\theta), \quad \phi(b, \theta) = 0 \)
and
(iii) \( \phi(a, \theta) = g(\theta), \quad \phi(b, \theta) = f(\theta) \),
where \(f(\theta)\) and \(g(\theta)\) are two periodic functions with Fourier series representations,
\[ f(\theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \quad \text{and} \quad g(\theta) = \frac{1}{2} g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta. \]

The solutions are
(i) \[ u(r, \theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \left( \frac{r^n - a^{2n} r^{-n}}{b^n - a^{2n} b^{-n}} \right) \]
(ii) \[ u(r, \theta) = \frac{1}{2} g_0 \frac{\ln(r/b)}{\ln(a/b)} + \sum_{n=1}^{\infty} g_n \cos n\theta \left( \frac{r^n - b^{2n} r^{-n}}{a^n - b^{2n} a^{-n}} \right) \]
The solution to (iii) is just the sum of the two solutions above.
3. Consider the heat equation with a source term, \( q(x,t) \):

\[
  u_t = u_{xx} + q.
\]

The boundary and initial conditions are

\[
  u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).
\]

Expand \( u, f \) and \( q \) in terms of Fourier sine series (with time-dependent coefficients in the cases of \( u(x,t) \) and \( q(x,t) \)), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

\[
  (i) \quad q = x(L-x), \quad f = 0,
\]

\[
  (ii) \quad q = x(L-x)e^{-t}, \quad f = 0
\]

and

\[
  (iii) \quad q = x(L-x)e^{-t}, \quad f = \sin\left(\frac{2\pi x}{L}\right).
\]

First,

\[
  x(L-x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)
\]

with (using integration by parts)

\[
  b_n = \frac{4L^2}{n^3\pi^3}\left[1 - (-1)^n\right].
\]

Now set

\[
  u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad \& \quad q(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin\left(\frac{n\pi x}{L}\right)
\]

allowing the PDE to be broken down into the system of ODEs,

\[
  \dot{B}_n + \frac{n^2\pi^2}{L^2}B_n = Q_n,
\]

subject to the initial conditions \( B_n(0) = 0 \) in parts \( (i) \) and \( (ii) \), and \( B_2(0) = 1 \) and all other \( B_n(0) = 0 \) in part \( (iii) \). The solutions are

\[
  (i) \quad B_n = \frac{L^2b_n}{n^2\pi^2}\left(1 - e^{-n^2\pi^2t/L^2}\right) \quad (Q_n = b_n)
\]

\[
  (ii) \quad B_n = \frac{L^2b_n}{n^2\pi^2 - 1}\left(e^{-t} - e^{-n^2\pi^2t/L^2}\right) \quad (Q_n = b_ne^{-t}),
\]

with \( B_n(t) = tb_ne^{-t} \) if \( n\pi = L \), and

\[
  (iii) \quad B_n = B_n^{(ii)} \text{ if } n \neq 2 \quad \& \quad B_2 = B_2^{(ii)} + e^{-4\pi^2t/L^2},
\]

where \( B_n^{(ii)} \) denotes the solution to part \( (ii) \).
And more

1. Solve the heat equation,

\[ u_t = u_{xx}, \]

for \(0 \leq x \leq \pi\) and \(t \geq 0\), subject to the boundary conditions,

\[ u_x(0, t) = u(\pi, t) = 0,\]

and the initial condition,

\[ u(x, 0) = x(\pi - x) \sin x.\]

Briefly justify your solution using Fourier series theory. The MATLAB code pde400-1.m provides a numerical solution to this problem using an in-built solver function PDEPE. Compare the numerical solution with your series solution, truncated to include only the first five terms, at the times and positions plotted by pde400-1.m.

The solution takes the form of a cosine series:

\[ u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\left(n - \frac{1}{2}\right)^2 t} \cos(n - \frac{1}{2}) \]

with (using integration by parts)

\[ a_n = \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin x \cos(n - \frac{1}{2}) x \ dx = \frac{(-1)^{n+1}}{(n + \frac{1}{2})^2} - \frac{(-1)^{n+1}}{(n - \frac{3}{2})^2} + \frac{2}{\pi(n + \frac{1}{2})^3} - \frac{2}{\pi(n - \frac{5}{2})^3}.\]

This is justified by Fourier series theory by extending \(u(x, t)\) and \(f(x)\) into \(\pi < x < 2\pi\) by \(f(x) = -f(2\pi - x)\) and \(u(x, t) = -u(2\pi - x, t)\), and then as even functions into \(-2\pi < x < 0\). This implies they can be represented as a Fourier cosine series, without the constant terms and with a sum over only the odd integers, for a period of \(2L = 4\pi\). The code in pde400-1x.m (which adds to pde400-1.m) compares the truncated sum with the numerical solution.

Figure 3: Output of pde400-1x.m
2. Solve the PDE
\[ \frac{\partial}{\partial r} \left( r^3 \frac{\partial \phi}{\partial r} \right) + 4r \frac{\partial^2 \phi}{\partial \theta^2} + r \phi = 0, \]
for \( 1 < r < R \) and \( 0 \leq \theta \leq 2\pi \), subject to the boundary conditions that \( u(r, \theta) \) is periodic in \( \theta \) and

(i) \( \phi(1, \theta) = 0, \quad \phi(R, \theta) = g(\theta), \)

(ii) \( \phi(1, \theta) = f(\theta), \quad \phi(R, \theta) = 0 \)

and

(iii) \( \phi(1, \theta) = f(\theta), \quad \phi(R, \theta) = g(\theta), \)

where \( f(\theta) \) and \( g(\theta) \) are two periodic functions with Fourier series representations,

\[ f(\theta) = \frac{1}{2} f_0 + \sum_{n=1}^{\infty} f_n \cos n\theta \quad \text{and} \quad g(\theta) = \sum_{n=1}^{\infty} g_n \sin n\theta. \]

Finally, resolve the PDE for \( R \to \infty \), with \( \phi(1, \theta) = F(\theta) \) prescribed and \( r \phi \) bounded for \( r \to \infty \), expressing your answer in terms of an integral involving \( F(\theta) \).

The solutions are

(i) \( u(r, \theta) = \frac{R}{r} \sum_{n=1}^{\infty} g_n \sin n\theta \left( \frac{r^{2n} - r^{-2n}}{R^{2n} - R^{-2n}} \right) \)

(ii) \( u(r, \theta) = \frac{1}{2} f_0 \log \left( \frac{R}{r} \right) + \frac{1}{r} \sum_{n=1}^{\infty} f_n \cos n\theta \left( \frac{R^{4n}r^{-2n} - r^{2n}}{R^{4n} - 1} \right) \)

The solution to (iii) is just the sum of the two solutions above. For the last part, we have

\[ \phi = \frac{1}{2} a_0 r^{-1} + \sum_{n=1}^{\infty} \left( r^{-2n-1} \left( a_n \cos n\theta + b_n \sin n\theta \right) \right), \]

where \( a_0, a_n \) and \( b_n \) are the coefficients of the Fourier series for \( F(\theta) \). Using the same manipulations as for warm-up problem 2, we see that

\[ u(r, \theta) = \frac{1}{2 \pi r} \int_{0}^{2\pi} \frac{(r^4 - 1)f(\hat{\theta})d\hat{\theta}}{r^4 + 1 - 2r^2 \cos(\hat{\theta} - \theta)}. \]

3. Consider the heat equation with a source term, \( q(x,t) \):

\[ u_t = u_{xx} + q. \]

The boundary and initial conditions are

\[ u(0, t) = u(\pi, t) = 0, \quad u(x, 0) = f(x). \]

Expand \( u, f \) and \( q \) in terms of Fourier sine series (with time-dependent coefficients in the cases of \( u(x,t) \) and \( q(x,t) \)), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for

(i) \( q = x(\pi - x), \quad f = 0, \)

(ii) \( q = x(\pi - x) \sin t, \quad f = 0 \)
and

\[ \text{(iii)} \quad q = x(\pi - x) \sin t, \quad f = x(\pi - x). \]

First,

\[ x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx \]

with (using integration by parts)

\[ b_n = \frac{4}{n^3 \pi} \left[ 1 - (-1)^n \right]. \]

Now set

\[ u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx \quad \& \quad q(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin nx, \]

allowing the PDE to be broken down into the system of ODEs,

\[ \dot{B}_n + n^2 B_n = Q_n, \]

subject to the initial conditions \( B_n(0) = 0 \) in parts (i) and (ii), and \( B_n(0) = b_n \) in part (iii). The solutions are

\[ (i) \quad B_n = \frac{b_n}{n^2} \left( 1 - e^{-n^2 t} \right) \quad (Q_n = b_n) \]

\[ (ii) \quad B_n = \frac{b_n}{1 + n^4} \left( e^{-n^2 t} - \cos t + n^2 \sin t \right) \quad (Q_n = b_n \sin t), \]

and

\[ (iii) \quad B_n = B_n^{(ii)} + b_n e^{-n^2 t}, \]

where \( B_n^{(ii)} \) denotes the solution to part (ii).