Coursework 1: Separation of Variables

Hand in solutions to the questions on page 1 only; pages 2 and 3 contain supplementary “warm-up” problems to practice on for your own enjoyment.

1. Solve the heat equation,
   \[ u_t = u_{xx}, \]
   for \( 0 \leq x \leq \pi \) and \( t \geq 0 \), subject to the boundary conditions,
   \[ u_x(0, t) = u_x(\pi, t) = 0, \]
   and the initial condition,
   \[ u(x, 0) = e^{-x}. \]

2. Solve Laplace’s equation,
   \[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0, \]
   inside the annulus, \( a \leq r \leq b \), subject to the boundary conditions,
   \[(i) \quad \phi(a, \theta) = 0, \quad \phi(b, \theta) = f(\theta), \]
   \[(ii) \quad \phi(a, \theta) = g(\theta), \quad \phi(b, \theta) = 0 \]
   and
   \[(iii) \quad \phi(a, \theta) = g(\theta), \quad \phi(b, \theta) = f(\theta), \]
   where \( f(\theta) \) and \( g(\theta) \) are two periodic functions with Fourier series representations,
   \[ f(\theta) = \sum_{n=1}^{\infty} f_n \sin n \theta \quad \text{and} \quad g(\theta) = \frac{1}{2} g_0 + \sum_{n=1}^{\infty} g_n \cos n \theta. \]

3. Consider the heat equation with a source term, \( q(x, t) \):
   \[ u_t = u_{xx} + q. \]
   The boundary and initial conditions are
   \[ u(0, t) = u(L, t) = 0, \quad u(x, 0) = f(x). \]
   Expand \( u, f \) and \( q \) in terms of Fourier sine series (with time-dependent coefficients in the cases of \( u(x, t) \) and \( q(x, t) \)), to find a set of ODEs with suitable initial conditions that are equivalent to the original PDE. Solve these ODEs and hence reconstruct the solution to the PDE for
   \[(i) \quad q = x(L - x), \quad f = 0, \]
   \[(ii) \quad q = x(L - x)e^{-t}, \quad f = 0 \]
   and
   \[(iii) \quad q = x(L - x)e^{-t}, \quad f = \sin(2\pi x/L). \]
Warm-up problems

1. Solve
   \[ u_t = \alpha u + u_{xx}, \quad 0 \leq x \leq \pi, \quad u_x(0, t) = u_x(\pi, t) = 0, \]
   where \( \alpha \) is a constant parameter, subject to (a) \( u(x, 0) = \cos x \), and (b) \( u(x, 0) = f(x) \) for some prescribed function \( f(x) \).

   Using separation of variables, we have \( u(x, t) = X(x)T(t) \), with
   \[ X_{xx} = -\lambda X, \quad T_t = (\alpha - \lambda)T. \]
   Applying \( X_x(0) = X_x(\pi) = 0 \) implies
   \[ \lambda = n^2, \quad X = \cos nx, \quad n = 0, 1, 2, \ldots \]
   Hence,
   \[ u(x, t) = \frac{1}{2}a_0 e^{\alpha t} + \sum_{n=1}^{\infty} a_n e^{(\alpha - n^2)t} \cos nx. \]
   In (a), \( a_n = 0 \) for \( n \neq 1 \) and \( a_1 = 1 \), so \( u = e^{(\alpha-1)t} \cos x \). In (b) we extend \( f(x) \) as an even function to \( -\pi \leq x \leq \pi \), allowing us to write
   \[ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_j = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos jx \, dx. \]

2. Solve \( \nabla^2 u = 0 \) outside the unit disk, \( r \geq 1 \), subject to \( u(1, \theta) = f(\theta) \) and \( u \) bounded as \( r \to \infty \).

   The solution by separation of variables is
   \[ u(r, \theta) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} r^{-m} [a_m \cos m\theta + b_m \sin m\theta], \]
   after discarding the solutions \( r^m \) and \( \ln r \) which diverge for \( r \to \infty \). The Fourier series expression of \( f(\theta) \) furnishes the coefficients \( a_j \) and \( b_j \) as the usual integrals:
   \[ a_j = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \cos j\theta \, d\theta, \quad b_j = \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \sin j\theta \, d\theta. \]
   Introducing these integrals into the solution, using \( \cos(A - B) = \cos A \cos B + \sin A \sin B \) and \( \cos z = (e^{iz} + e^{-iz})/2 \), interchanging the order of the integral and the sum, and then summing the series using
   \[ \frac{x}{1-x} = \sum_{m=1}^{\infty} x^m, \]
   gives, with a little algebra,
   \[ u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{r^2 - 1) f(\theta) d\theta}{r^2 + 1 - 2r \cos(\theta - \theta)}. \]
3. Solve

\[ u_t = \alpha u - u_{xxxx}, \quad 0 \leq x \leq \pi \quad u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad u(x, 0) = f(x), \]

and then

\[ u_t = \alpha u - u_{xxxx} + g(x), \quad 0 \leq x \leq \pi \quad u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad u(x, 0) = f(x), \]

where \( \alpha \) is a constant, by expanding \( u(x, t) \) as a Fourier sine series with time-dependent coefficients, \( B_n(t) \).

For the homogeneous problem, we set

\[ u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin nx. \]

From the PDE,

\[ \sum_{n=1}^{\infty} \left( \dot{B}_n - \alpha B_n + n^4 B_n \right) \sin nx = 0. \]

Hence, the coefficients satisfy the ODEs

\[ \dot{B}_n = \alpha B_n - n^4 B_n. \]

The initial condition can be extended to \( -\pi \leq x \leq \pi \) as an odd function, and \( f(x) \) therefore written as

\[ f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx. \]

Hence,

\[ B_n(t) = b_n e^{(\alpha - n^4)t}. \]

In the inhomogeneous case, we extend \( g(x) \) to \( -\pi \leq x \leq \pi \) as an odd function:

\[ g(x) = \sum_{n=1}^{\infty} g_n \sin nx. \]

The ODEs extracted from the PDE are now:

\[ \dot{B}_n = \alpha B_n - n^4 B_n + g_n. \]

The solutions, subject to \( B_n(0) = b_n \) are

\[ B_n(t) = (b_n - G_n) e^{(\alpha - n^4)t} + G_n, \quad G_n = \frac{g_n}{n^4 - \alpha}, \]

provided \( \alpha^{1/4} \) is not an integer. If \( \alpha = N^4 \), for some integer \( N \), then the solution above remains valid for \( n \neq N \); for \( n = N \), the ODE is

\[ \dot{B}_N = g_N \quad \rightarrow \quad B_N = b_N + g_N t. \]
Actual solutions

1. (3 marks) The solution takes the form of a Fourier cosine series:

\[ u(x, t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx \]

with (using integration by parts)

\[ a_0 = \frac{2}{\pi} \int_0^\pi e^{-x} dx = \frac{2}{\pi} (1 - e^{-\pi}) \]
\[ a_n = \frac{2}{\pi} \int_0^\pi e^{-x} \cos nx \, dx = \frac{2}{\pi} \frac{1 - (-1)^n e^{-\pi}}{n^2 + 1}. \]

2. (3 marks)

(i) \( u(r, \theta) = \sum_{n=1}^{\infty} f_n \sin n\theta \frac{(r^n - a^{2n} r^{-n})}{(b^n - a^{2n} b^{-n})} \)

(ii) \( u(r, \theta) = \frac{1}{2} g_0 \frac{\ln(r/b)}{\ln(a/b)} + \sum_{n=1}^{\infty} g_n \cos n\theta \frac{(r^n - b^{2n} r^{-n})}{(a^n - b^{2n} a^{-n})} \)

The solution to (iii) is just the sum of the two solutions above.

3. (10 marks) First,

\[ x(L - x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right) \]

with (using integration by parts)

\[ b_n = \frac{4L^2}{n^3 \pi^3} [1 - (-1)^n]. \]

Now set

\[ u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin \left( \frac{n\pi x}{L} \right) \quad \& \quad q(x, t) = \sum_{n=1}^{\infty} Q_n(t) \sin \left( \frac{n\pi x}{L} \right) \]

allowing the PDE to be broken down into the system of ODEs,

\[ \ddot{B}_n + \frac{n^2 \pi^2}{L^2} B_n = Q_n, \]

subject to the initial conditions \( B_n(0) = 0 \) in parts (i) and (ii), and \( B_2(0) = 1 \) and all other \( B_n(0) = 0 \) in part (iii). The solutions are

(i) \( B_n = \frac{L^2 b_n}{n^2 \pi^2} (1 - e^{-n^2 \pi^2 t/L^2}) \) \( (Q_n = b_n) \)

(ii) \( B_n = \frac{L^2 b_n}{n^2 \pi^2 - 1} (e^{-t} - e^{-n^2 \pi^2 t/L^2}) \) \( (Q_n = b_n e^{-t}) \)

with \( B_n(t) = b_n t \) if \( n\pi = L, \) and

(iii) \( B_n = B_n^{(ii)} \) if \( n \neq 2 \) \& \( B_2 = B_2^{(ii)} + e^{-4\pi^2 t/L^2}, \)

where \( B_n^{(ii)} \) denotes the solution to part (ii).