Surface tension and water waves

The surface tension of water is \( \sigma = 70 \text{ dynes/cm} \), and the Laplace-Young relation relates the pressure difference across the water surface to its curvature:

\[
p_{\text{fluid}} - p_{\text{atmos}} = -\frac{\sigma h_{xx}}{(1 + h_x^2)^{3/2}}.
\]

(1) Use dimensional analysis to formulate two dimensionless numbers that estimate the importance of surface tension to surface gravity waves in a layer of depth \( H \) with wavenumber \( k \), in either the shallow or deep water limits. Estimate the size of the relevant parameter for

(i) shallow ocean waves on the beach, with \( H \sim 10m \).

(ii) “deep” waves in your teacup, with wavelength, \( 2\pi/k \sim 1\text{cm} \).

(2) Derive a modified dispersion relation for incompressible, irrotational, inviscid water waves that incorporates surface tension via the Laplace-Young relation. Extract dimensionless numbers from this relation that confirm the numbers guessed at with dimensional analysis.
Shallow water seiches

This problem surrounds the shallow water equations, linearized about a motionless equilibrium with depth $H(x)$:

$$ u_t = -gh_x, \quad h_t + (H u)_x = 0. $$

(1) The equilibrium is a uniform tank of length $L$ and depth $H = constant$. The perturbations satisfy $u(0,t) = u(L,t) = 0$.

The practical part: Find a suitable container. Pour in a layer of water. Tilt the tank to excite a seiche. Measure the length and the depth of the water reservoir, and observe the period of the seiche.

The theoretical part: Calculate what the period should be theoretically by searching for separable solutions of the linearized shallow water equations, $[u(x,t), h(x,t)] = [u(x), h(x)] e^{-i \omega t}$, where $\omega$ is the frequency (giving a period of $2\pi/\omega$).

(2) The equilibrium is a lake with $H(x) = H_0(1 - x^2/L^2)$. Look for separable solutions of the shallow water equations once again, and set up a differential equation for $\dot{h}(x)$. The solutions should be regular, even at the lake’s edges, $x = \pm L$. Find the normal-mode frequencies, $\omega$. (You might want to recall/research Legendre’s equation.)
Topographic beta-effect
Consider a uniform flow on the $f$–plane with weakly varying depth:

$$u = U \ll 1,$$
$$v = 0,$$
$$h = H(y) = H_0 + H_1(y), \quad |H_1| \ll H_0.$$  

What is the relation between $U$ and $H_1(y)$? Write down the linear equations for small perturbations about this equilibrium. Take the curl of the momentum equations and then use the mass conservation equation to derive an equation of the form,

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q + Q_y v = 0,$$

for some $q$ and $Q$. How is this relation related to conservation of potential vorticity, expressed in general as

$$\frac{D}{Dt} \left( \frac{v_x - u_y - f}{h} \right) = 0?$$

Make the geostrophic approximation suitable for slow waves to derive a dispersion relation for topographically modified Rossby waves. Use the results to explain you might set up a laboratory experiment to model ocean circulation.
Internal gravity waves

Consider a rotating fluid layer in which there is a background density gradient, $\beta$. Two-dimensional linear perturbations satisfy the equations:

$$
\begin{align*}
    u_t - fv &= - \frac{p_x}{\rho_0} \\
    v_t + fu &= 0 \\
    w_t &= - \frac{p_z}{\rho_0} - \frac{\rho' g}{\rho_0} \\
    u_x + w_z &= 0 \\
    \rho'_t - \frac{N^2}{g} w &= 0,
\end{align*}
$$

where $N^2$ (the square of the buoyancy frequency) is proportional to $|\beta|$, $\rho_0$ is a constant reference density, and $\rho'(x, z, t)$ is the density perturbation. Briefly comment on what has been done to the governing fluid equations to arrive at this set of equations.

Look for solutions with the dependence, $\exp i (k \cdot x - \omega t)$, where

$$
k \equiv (k, m) \equiv K (\cos \theta, \sin \theta)
$$

is the wavenumber vector, and $\theta$ is the angle of phase propagation with respect to the horizontal. Derive the dispersion relation, $\omega = \omega(k, m)$. Hence show that phase of internal gravity waves always propagates with the same angle with respect to the vertical given the frequency and background fluid properties, whatever the wavenumber $K$. What is that angle? What frequencies are allowed for the waves? Comment on what this signifies for wave reflection from an inclined wall.

Show that the group velocity of internal gravity waves, $c_g = \partial \omega / \partial k$, is orthogonal to the phase velocity ($i.e.$ $k$). What distinctive features of a wave packet of internal gravity waves does this imply?
The Richardson Number Criterion (Howard, 1961)

Shear flow in a fluid that has a density gradient ("stratified shear flow") is governed approximately by the equations,

\begin{align*}
    u_x + w_z &= 0 \\
    \rho(u + uu_x + uu_z) &= -px \\
    \rho(w + uw_x + ww_z) &= -pz - g\Theta \\
    \Theta_t + u\Theta_x + w\Theta_z &= 0,
\end{align*}

where \( \rho \) is a constant reference density and \( \Theta \) is the variation in density that is only accounted for in the gravity term (the "Boussinesq approximation").

Show that there is a basic state with \( u = U(z), p = P(z) \) and \( \Theta = -\beta z \), where \( \beta \) is a constant density gradient. How are \( U, P \) and \( \beta \) related?

Consider linear perturbations about the equilibrium. By introducing a streamfunction, \( \psi(x, z, t) \), and adopting the dependence, \( \psi = \Psi(z) \exp[ik(x - ct)] \), derive an ordinary differential equation (ODE) for \( \Psi \) similar to Rayleigh’s equation for incompressible, inviscid shear flow.

Let \( \Psi = (U - c)^{1/2}G(z) \). Rewrite the ODE in terms of \( G \). Multiply the new equation by \( G^* \), extract the imaginary part, and thereby show that the flow cannot be unstable if \( Ri > 1/4 \) where

\[ Ri = \frac{g\beta}{\rho U_z^2} \]

is the "Richardson number".