Algebraic problems

1. Find the rescalings for the roots of

\[ \epsilon^4 x^3 - (1 + 2\epsilon + 5\epsilon^4)x^2 + (5 + 10\epsilon + 6\epsilon^4 + \epsilon^5 - \epsilon^6)x - 6 - 13\epsilon - \epsilon^2 + 2\epsilon^3 = 0, \]

and hence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

\[ \delta x^3 - (1 + 5\delta + 2\delta^2 + \delta^3)x^2 + (3 + 8\delta + 14\delta^2 + 2\delta^3 - \delta^4)x - 9\delta - 18\delta^2 + \delta^3 + 2\delta^4 = 0, \]

for \( \delta \ll 1 \) and \( \delta \gg 1 \).

3. Develop perturbation solutions to

\[ x^3 - (6 + 3\epsilon + 2\epsilon^2)x^2 + (12 + 11\epsilon + 11\epsilon^2 + 4\epsilon^3)x - 8 - 10\epsilon - 13\epsilon^2 - 7\epsilon^3 - 2\epsilon^4 = 0 \]

finding the three terms in the approximation for each root, \( x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha}x_{2\alpha} \), and determining \( \alpha \) along the way.

4. Develop three terms of the perturbation solutions to the real roots of

\[ xe^{-x^3} = 2\epsilon, \]

identifying the scalings in the expansion sequence \( \delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + \ldots \)

5*. Find an algebraic equation for the frequencies of the normal modes of a string, satisfying

\[ \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad \left[ \frac{\partial u}{\partial x} - u \right]_{x=1} = 0. \]

Provide an asymptotic solution with three terms for these frequencies. \textit{i.e. pose the solution} \( u(x,t) = e^{i\omega t}U(x) \) and find an equation for \( \omega \) by solving the PDE and applying the boundary conditions; solve this equation using the asymptotic method of your choice. You will need to figure out what the small parameter should be.
Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem \( A\mathbf{x} = \lambda \mathbf{x} + \epsilon \mathbf{B}\mathbf{x} \), for

\[
(a) \quad \begin{pmatrix} 7 & -9 \\ -9 & -17 \end{pmatrix} \quad (b) \quad \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}
\]

and

\[
\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}
\]

2. Find the nontrivial corrections to the leading-order eigenvalues of the differential equation,

\[
y'' + \lambda [1 + \epsilon f(x)] y = 0,
\]

with \( y(0) = y(\pi) = 0 \), for (a) \( f(x) = x \) and (b*) \( f(x) = x - \frac{\pi}{2} \).

3. Normal modes of a slightly mis-shapen membrane

Normal-mode solutions to the wave equation \( \nabla^2 \phi = \phi_{tt} \) take the form \( \phi(x, y, t) = \Phi(x, y) \cos(\omega t) \) and therefore satisfy

\[
\Phi_{xx} + \Phi_{yy} = -\omega^2 \Phi,
\]

where subscripts denote partial derivatives. Consider a slightly mis-shapen membrane on the boundary of the domain,

\[
\epsilon \sin y \leq x \leq \pi, \quad 0 \leq y \leq \pi,
\]

with boundary conditions

\[
\Phi(x, 0) = \Phi(x, \pi) = 0 \quad \& \quad \Phi_x(\epsilon \sin y, y) = \Phi_x(\pi, y) = 0.
\]

Show that \( \Phi = \cos nx \sin y \) is a leading-order eigenfunction, with \( n \) an integer. Find the corresponding eigenvalue \( \omega \). Calculate the \( O(\epsilon) \) correction to the eigenvalue for (a) \( n = 1 \), (b) \( n = 0 \) and (c*) \( n = 2 \).