Algebraic problems

1. Find the rescalings for the roots of

$$\epsilon^3 x^3 - (1 - 2\epsilon + 3\epsilon^3 - 2\epsilon^4)x^2 + (3 - 8\epsilon + 4\epsilon^2 + 2\epsilon^3 - 8\epsilon^5)x - 2 + 4\epsilon + 8\epsilon^2 - 16\epsilon^3 = 0,$$

and thence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

$$\delta x^3 - (1 + \delta + \delta^2 - 2\delta^3)x^2 + (2 - \delta + \delta^2 - 4\delta^3 + 2\delta^4)x - 4\delta + 10\delta^2 - 8\delta^3 + 2\delta^4 = 0,$$

for $\delta \ll 1$ and $\delta \gg 1$.

3. Develop perturbation solutions to

$$8x^3 - (12 - 8\epsilon + 24\epsilon^2)x^2 + (6 - 8\epsilon + 24\epsilon^2 - 48\epsilon^3 + 24\epsilon^4)x - 1 + 2\epsilon - 6\epsilon^2 + 24\epsilon^3 - 44\epsilon^4 + 40\epsilon^5 - 8\epsilon^6 = 0$$

finding the three terms in the approximation for each root, $x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha}$, and determining α along the way.

4. Develop three terms of the perturbation solutions to the real roots of

$$e^{-\sqrt{x}} = 2\epsilon x,$$

identifying the scalings in the expansion sequence $\delta_0(\epsilon)x_0 + \delta_1(\epsilon)x_1 + \delta_2(\epsilon)x_2 + ...$

5*. Find an algebraic equation for the frequencies of the normal modes of a string, satisfying

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0, \quad \left[\frac{\partial u}{\partial x} - u\right]_{x=1} = 0.$$

Provide an asymptotic solution with three terms for these frequencies. *i.e. pose the solution $u(x,t) = e^{i\omega t}U(x)$ and find an equation for ω by solving the PDE and applying the boundary conditions; solve this equation using the asymptotic method of your choice. You will need to figure out what the small parameter should be.*

Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem $A\mathbf{x} = \lambda\mathbf{x} + \epsilon B\mathbf{x}$, for

$$(a) \quad \begin{pmatrix} 6 & 3 \\ 3 & 14 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 9 & -2 \\ 8 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

2. Find the nontrivial corrections to the leading-order eigenvalues of the differential equation,

$$y'' + \lambda y + \epsilon y f(x) = 0,$$

with $y(0) = y(\pi) = 0$, for (a) $f(x) = \sin x$ and **(b*)** $f(x) = \sin 2x$.

3. Normal modes of a slightly mis-shapen membrane

Normal-mode solutions to the wave equation $\nabla^2\phi = \phi_{tt}$ take the form $\phi(x, y, t) = \Phi(x, y)\cos(\omega t)$ and therefore satisfy

$$\Phi_{xx} + \Phi_{yy} = -\omega^2\Phi,$$

where subscripts denote partial derivatives. Consider a slightly mis-shapen membrane with with $\phi = 0$ on the boundary of the domain,

$$0 \le x \le \pi, \quad \epsilon x(\pi - x) \le y \le \pi.$$

Show that $\Phi = \sin x \sin ny$ is a leading-order eigenfunction, with n an integer. Find the corresponding eigenvalue ω. Calculate the $O(\epsilon)$ correction to the eigenvalue for (a) $n = 1$ and **(b*)** $n = 2$.

Integrals

1. Use the method of repeated integration by parts or rescaling to obtain five terms in the asymptotic approximation to the integral,

$$\int_x^\infty t^{-5} e^{-t^2}\, dt,$$

for $x \to 0$. Note that

$$\gamma = -\int_0^\infty e^{-u} \log u \; du$$

is Euler's constant.

2. Find the leading-order behaviour for $x \gg 1$ of

$$(a) \int_0^\infty e^{xt^3(5-3t^2)} \frac{t^2 dt}{(1+t^2)} \qquad (b) \int_{-\pi}^{\pi} t^2 e^{-x \sin^2 t} dt \qquad (c) \int_0^\pi \sqrt{\sinh t}\; e^{-x \sinh^5 t} dt$$

3. Evaluate the first two terms as $\epsilon \to 0$ of

$$\int_0^\infty \frac{dx}{(\epsilon + x^2)^{1/2}(1+x^2)}$$

4. Evaluate the first two terms as m approaches unity from below of

$$\int_{-\pi/4}^{\pi/4} (1 - m^2 \tan^2 \theta)^{-1/2} d\theta$$

5.* Evaluate the first two terms as $\epsilon \to 0$ of

$$\mathcal{N}(z) = \int_z^1 \frac{\sin x}{\epsilon + x^2} dx$$

where z is a real parameter that is $O(1)$ or smaller and not necessarily positive.

Matched asymptotic expansions

1. Consider

$$\epsilon y'' + (x+1)y' + y + \epsilon y = 2x, \qquad \text{in } 0 \le x \le 1,$$

with $y(0) = 0$ and $y(1) = \sin\epsilon$. Find three terms of the outer solution, applying only the boundary condition at $x = 1$. Next find three terms in an inner approximation for the boundary layer near $x = 0$ applying the boundary condition at $x = 0$. Determine the constants of integration by matching (a) over an intermediate region, and (b) using van Dyke's rule with $P = Q = 2$. Compute the composite approximation, $C_{2,2}y$.

2. The function $y(x)$ satisfies

$$\epsilon x^p y'' + \sigma y' + y = 0, \qquad \text{in } 0 \le x \le 1,$$

for $p < 1$, $y(0) = 0$ and $y(1) = 1$.
(a) For $\sigma = +1$, first find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.
(b) Now find the leading-order solution for $\sigma = -1$.

3. Calculate two terms of the outer solution of

$$y' = 2\epsilon\frac{y^3}{x^3}\left[1 + \epsilon y + \frac{3\epsilon}{y} + \epsilon x(y^2 - 1)\right] \qquad \text{in } 0 \le x \le 1,$$

with $y(1) = 1$. Locate the non-uniformity of the asymptoticness and hence the rescaling for an inner region. Thence find two terms for this inner solution. Is there another boundary layer nested inside the inner region, and if so what is the leading-order solution there?

4. The function $f(r)$ satisfies the equation,

$$\frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} + \epsilon(1+f)\frac{df}{dr} = 0,$$

in $r \ge 1$, with $\epsilon > 0$ and the boundary conditions, $f = 0$ on $r = 1$ and $f \to 1$ as $r \to \infty$. Obtain an asymptotic expansion for f at fixed r as $\epsilon \to 0$ in the asymptotic sequence, 1, ϵ, $\epsilon^2 \log(1/\epsilon)$, ϵ^2, ... Then find an expansion for f at fixed $\rho = \epsilon r$ as $\epsilon \to 0$ in the sequence, 1, ϵ^2, ... Match these expressions.

5*. The function $f(x)$ satisfies

$$f_{xx} - \frac{4}{x^3}\epsilon^2 f^2 f_x = 0, \qquad f(0) = 1, \qquad f(1) = 0.$$

Obtain an asymptotic expansion for f at fixed $\xi = x\epsilon^{-\alpha}$ as $\epsilon \to 0$ for some α (that you should determine), in the sequence $1 + \epsilon g_1(\xi) + \epsilon^2 g_2(\xi)$. Match this expansion to a second solution for f at fixed x and $\epsilon \to 0$, that includes terms upto and including $O(\epsilon^2)$.

Multiple Scales

1. Obtain an asymptotic approximation for x to order one, which is valid for $t = O(\epsilon^{-1})$, when

$$\ddot{x} + \epsilon(\dot{x} + x)^3 + x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$$

2. Find the leading-order approximation for times of order ϵ^{-1} to

$$\ddot{x} + x - y^2 + \epsilon y\dot{x} = 0, \qquad \dot{y} + \epsilon x\dot{x} = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 0, \quad y(0) = 1.$$

3. Obtain an asymptotic approximation for x to order one, which is valid for $t = O(\epsilon^{-1})$, when

$$\ddot{x} + x + \epsilon\dot{x}|x| = 0, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

4. Mathieu problem:

$$\ddot{y} + (a + 2\epsilon \sin t)y = 0, \qquad a = \frac{1}{4}n^2 + \epsilon a_1 + \epsilon^2 a_2.$$

We are interested in discovering when solutions grow exponentially; the initial conditions are not particularly relevant. Let's solve the problem with multiple scales:

$$y = y_0(\tau, T, T_2) + \epsilon y_1(\tau, T, T_2) + \epsilon^2 y_2(\tau, T, T_2) + ... \qquad \tau = t, \ T = \epsilon t, \ T_2 = \epsilon^2 t.$$

For $n = 1$, show that some of the inhomogeneous terms of the y_1 equation are resonant. Eliminate the baddies, and show that the solution grows exponentially (signifying parametric instability) over a certain range of a_1; translate this back to a sector of the $(a, \epsilon)-$plane. Repeat the exercise for $n = 2$, convincing yourself that we should drop a_1 and the first slow timescale T. Last, attack the case $n = 0$.

5*. Use the method of multiple scales to solve the pendulum problem

$$\ddot{\theta} + \sin\theta = \epsilon\dot{\theta}(\alpha - \dot{\theta}^2) \qquad \theta(0) = \pi/2, \quad \dot{\theta} = 0.$$

Hence determine the long-time fate of the pendulum, and derive a criterion that dictates whether it oscillates back and forth, with $|\theta| < \pi$, or continually circulates around. *The elliptic integrals $E(k)$ and $K(k)$ might be useful.*