Algebraic problems

1. Find the rescalings for the roots of

$$\epsilon^2 x^3 - (1 + 3\epsilon - 3\epsilon^2 - \epsilon^3)x^2 + (3 - 8\epsilon - \epsilon^2 - \epsilon^3 - 6\epsilon^4)x - 2 + 7\epsilon + 3\epsilon^2 - 18\epsilon^3 = 0,$$

and thence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

$$\delta x^3 - (1 - \delta + 3\delta^2 - \delta^3)x^2 + (1 + \delta - 5\delta^2 + 3\delta^3 - \delta^4)x - 2\delta + 3\delta^2 + 3\delta^3 - 2\delta^4 = 0,$$

for $\delta \ll 1$ and $\delta \gg 1$.

3. Develop perturbation solutions to

$$x^3 - (9 + 3\epsilon + 2\epsilon^2)x^2 + (27 + 17\epsilon + 12\epsilon^2 + 8\epsilon^3)x - 27 - 24\epsilon - 19\epsilon^2 - 18\epsilon^3 - 8\epsilon^4 = 0$$

finding the three terms in the approximation for each root, $x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha}$, and determining α along the way.

4. Develop three terms of the perturbation solutions to the real roots of

$$e^{x - x^2} = 2\epsilon x.$$

Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem $A\mathbf{x} = \lambda\mathbf{x} + \epsilon B\mathbf{x}$, for

$$(a) \quad \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 8 & 4 \\ -4 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

2. Find the corrections to the leading-order eigenvalues of the differential equation,

$$y'' + \lambda y + \epsilon e^x y = 0,$$

with $y(-1) = y(1) = 0$.

3. Normal modes of a slightly mis-shapen membrane

Consider the two-dimensional wave equation,

$$\phi_{xx} + \phi_{yy} = \phi_{tt},$$

where subscripts denote partial derivatives, on the region,

$$0 \le x \le 1, \quad \epsilon x \sin \pi x \le y \le 1,$$

with $\phi = 0$ on the boundary. Consider normal-mode solutions for which $\phi(x, y, t) = \Phi(x, y)\cos(\omega t)$. Show that $\Phi = \sin \pi x \sin 2\pi y$ and $\omega = \pi\sqrt{5}$ is a leading-order eigensolution. Calculate the $O(\epsilon)$ corrections to this solution.

Integrals

1. Use the method of repeated integration by parts or rescaling to obtain four terms in the asymptotic approximation to the integral,

$$\int_x^\infty t^{-3} e^{-t/2} dt,$$

for $x \to 0$. Note that

$$\gamma = -\int_0^\infty e^{-t} \log t \, dt$$

is Euler's constant.

2. Find the leading-order behaviour for $x \gg 1$ of

$$(a) \int_0^\infty e^{xt(5-t^4)} \frac{dt}{(1+3t^3)} \qquad (b) \int_{-\infty}^\infty e^{t - x \sinh^2 t} dt \qquad (c) \int_0^\infty \sqrt{\sin t} \, e^{-x \sinh^4 t} dt$$

3. Evaluate the first two terms as $\epsilon \to 0$ of

$$\int_0^\infty \frac{dx}{(\epsilon + x)^{5/2}(1+x)}$$

4. Evaluate the first two terms as m approaches unity from below of

$$\int_0^{\pi/2} \frac{\sin^2 \theta}{(1 - m^2 \sin^2 \theta)^{1/2}} d\theta$$

Matched asymptotic expansions

1. Consider

$$\epsilon y'' + (1-\epsilon)y' + y = 0, \qquad \text{in } 0 \le x \le 1,$$

with $y(0) = 0$ and $y(1) = e^{-1}$. Find three terms of the outer solution, applying only the boundary condition at $x = 1$. Next find three terms in an inner approximation for the boundary layer near $x = 0$ applying the boundary condition at $x = 0$. Determine the constants of integration by matching (a) over an intermediate region, and (b) using van Dyke's rule with $P = Q = 2$. Compute the composite approximation, $C_{2,2}y$.

2. The function $y(x)$ satisfies

$$\epsilon y'' + \sigma x^q y' + y = 0, \qquad \text{in } 0 \le x \le 1,$$

for $0 < q < 1$, $\sigma = +1$, and $y(0) = 0$ and $y(1) = 1$. First find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations. Comment on the case $q > 1$ and $\sigma = \pm 1$.

3. Calculate two terms of the outer solution of

$$(1+\epsilon)x^2 y' = \epsilon[(1-\epsilon)xy^2 - (1+\epsilon)x^3 + y^3 - \epsilon y^2] \qquad \text{in } 0 \le x \le 1,$$

with $y(1) = 1$. Locate the non-uniformity of the asymptoticness and hence the rescaling for an inner region. Thence find two terms for this inner solution.

3. The function $f(r)$ satisfies

$$f_{rr} + \frac{3}{2r} f_r + 2\epsilon f^2 f_r = 0, \qquad \text{in } r > 1,$$

and is subject to the boundary conditions, $f(1) = 0$ and $f \to 1$ as $r \to \infty$. Obtain an asymptotic expansion for f at fixed r and $\epsilon \to 0$ in the asymptotic sequence, 1, $\epsilon^{1/2}$, $\epsilon \log(1/\epsilon)$, ϵ. Then find an expansion for f at fixed $\rho = \epsilon r$ as $\epsilon \to 0$ in the sequence 1, $\epsilon^{1/2}$, ϵ. Match these expansions.

Multiple Scales

1. Obtain an asymptotic approximation for x to order one, which is valid for $t = O(\epsilon^{-1})$, when

$$\ddot{x} + \epsilon[(\dot{x}^2 - 1)\dot{x} + x\dot{x}^2] + x = 0$$

with $x(0) = 0$ and $\dot{x}(0) = 1$.

2. Find the leading-order approximation for times of order ϵ^{-1} to

$$\ddot{x} + \epsilon y\dot{x} + x - \epsilon x^3 = y^3$$

$$\dot{y} = \epsilon(x - y^3)$$

with $\dot{x}(0) = 0$, $y(0) = 1$ and $x(0) = 2$.

3. Obtain an asymptotic approximation for x to order one, which is valid for $t = O(\epsilon^{-1})$, when

$$\ddot{x} + x = \epsilon\frac{x}{|x|}, \quad x(0) = 1, \quad \dot{x}(0) = 0.$$

4. Mathieu problem. The equation:

$$\ddot{y} + (a + 2\epsilon\cos t)y = 0$$

with

$$a = \frac{1}{4}n^2 + \epsilon a_1 + \epsilon^2 a_2$$

We are interested in discovering when solutions grow exponentially; the initial conditions are not particularly relevant. Let's solve the problem with multiple scales:

$$y = y_0(\tau, T, T_2) + \epsilon y_1(\tau, T, T_2) + \epsilon^2 y_2(\tau, T, T_2) + ... \qquad \tau = t,\ T = \epsilon t,\ T_2 = \epsilon^2 t.$$

Verify that the leading-order solution can be written as

$$y_0 = A(T, T_2)\cos\frac{n\tau}{2} + B(T, T_2)\sin\frac{n\tau}{2}$$

For $n = 1$, show that some of the inhomogeneous terms of the y_1 equation are resonant. Eliminate the baddies, write a single equation for A, and show that the solution grows exponentially (signifying parametric instability) over a certain range of a_1; translate this back to a sector of the $(a, \epsilon)-$plane.

For $n = 2$, convince yourself that we should drop the first slow timescale T, and write

$$y_0 = A(T_2)\cos\tau + B(T_2)\sin\tau$$

Now press on to $O(\epsilon^2)$ and eliminate the bad guys again, to write the equation for A. Once more, determine the sector of the $(a, \epsilon)-$plane where there is exponential growth.

Finally, repeat the exercise for $n = 0$.