Algebraic problems

1. Find the rescalings for the roots of

\[ \epsilon^2 x^3 - (1 + 3\epsilon - 3\epsilon^2 - \epsilon^3)x^2 + (3 - 8\epsilon - \epsilon^2 - \epsilon^3 - 6\epsilon^4)x - 2 + 7\epsilon + 3\epsilon^2 - 18\epsilon^3 = 0, \]

and hence find two (non-trivial) terms in the approximation for each root, using (a) iteration and (b) expansion.

2. Develop two terms of the perturbation solutions to

\[ \delta x^3 - (1 - \delta + 3\delta^2 - \delta^3)x^2 + (1 + \delta - 5\delta^2 + 3\delta^3 - \delta^4)x - 2\delta + 3\delta^2 + 3\delta^3 - 2\delta^4 = 0, \]

for \( \delta \ll 1 \) and \( \delta \gg 1 \).

3. Develop perturbation solutions to

\[ x^3 - (9 + 3\epsilon + 2\epsilon^2)x^2 + (27 + 17\epsilon + 12\epsilon^2 + 8\epsilon^3)x - 27 - 24\epsilon - 19\epsilon^2 - 18\epsilon^3 - 8\epsilon^4 = 0 \]

finding the three terms in the approximation for each root, \( x = x_0 + \epsilon^\alpha x_\alpha + \epsilon^{2\alpha} x_{2\alpha} \), and determining \( \alpha \) along the way.

4. Develop three terms of the perturbation solutions to the real roots of

\[ e^{x-x^2} = 2\epsilon x. \]
Eigenproblems and regularly perturbed differential equations

1. Find the corrections to the leading-order eigenvalues of the matrix problem $Ax = \lambda x + \epsilon Bx$, for

   \[(a) \quad \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad (b) \quad \begin{pmatrix} 8 & 4 \\ -4 & 0 \end{pmatrix}\]

   and

   \[B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\]

2. Find the corrections to the leading-order eigenvalues of the differential equation,

   \[y'' + \lambda y + \epsilon e^x y = 0,\]

   with $y(-1) = y(1) = 0$.

3. **Normal modes of a slightly mis-shapen membrane**

   Consider the two-dimensional wave equation,

   \[\phi_{xx} + \phi_{yy} = \phi_{tt},\]

   where subscripts denote partial derivatives, on the region,

   \[0 \leq x \leq 1, \quad \epsilon x \sin \pi x \leq y \leq 1,\]

   with $\phi = 0$ on the boundary. Consider normal-mode solutions for which $\phi(x, y, t) = \Phi(x, y) \cos(\omega t)$. Show that $\Phi = \sin \pi x \sin 2\pi y$ and $\omega = \pi \sqrt{5}$ is a leading-order eigensolution. Calculate the $O(\epsilon)$ corrections to this solution.