Some Notes on Taylor Polynomials and Taylor Series

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UBC's courses MATH 100/180 and MATH 101 introduce students to the ideas of Taylor polynomials and Taylor series in a fairly limited way. In these notes, we present these ideas in a condensed format. For students who wish to gain a deeper understanding of these concepts, we encourage a thorough reading of the chapter on Infinite Sequences and Series in the accompanying text by James Stewart.

1 Taylor Polynomials

We have considered a wide range of functions as we have explored calculus. The most basic functions have been the polynomial functions like

$$p(x) = x^3 + 9x^2 - 3x + 2,$$

which have particularly easy rules for computing their derivatives. As well, polynomials are evaluated using the simple operations of multiplication and addition, so it is relatively easy to compute their exact values given x. On the other hand, the transcendental functions (e.g., $\sin x$ or $\ln x$) are more difficult to compute. (Though it seems trivial to evaluate $\ln(1.75)$, say, by pressing a few buttons on your calculator, what really happens when you push the **ln** button is a bit more involved.)

Question: Is it possible to *approximate* a given function by a polynomial? That is, can we find a polynomial of a given degree n that can be substituted in place of a more complex function without too much error?

We have already considered this question in the specific case of *linear approximation*. There we took a specific function, f, and a specific point on the graph of that function, $(a, f(a))$, and approximated the function near $x = a$ by its tangent line at $x = a$. Explicitly, we approximated the curve

$$y = f(x)$$

by the straight line

$$y = L(x) = f(a) + f'(a)(x - a).$$

In doing this, we used two pieces of data about the function f at $x = a$ to construct this line: the function value, $f(a)$, and the function's derivative at $x = a$, $f'(a)$. Thus, this approximating linear function agrees *exactly* with f at $x = a$ in that $L(a) = f(a)$ and $L'(a) = f'(a)$. Of course, as we move away from $x = a$, in general the graph $y = f(x)$ deviates from the tangent line, so there is some error in replacing $f(x)$ by its linear approximant $L(x)$. We will discuss this error quantitatively in the next section.

Now, consider how we might construct a polynomial that is a good approximation to $y = f(x)$ near $x = a$. Straight lines, graphs of polynomials of degree one, do not curve, but we know that the graphs of quadratics (the familiar parabolas), cubics, and other higher degree polynomials have graphs that do curve. So, the question becomes: How do we find the coefficients of a polynomial of degree n so that it well approximates a given function f near a point given by $x = a$?

Let us start with an example. Consider the exponential function, $f(x) = e^x$, near $x = 0$. We have already found that

$$e^x \approx 1 + x$$

by considering its linear approximation at $x = 0$. From the graphs of these two function, we know that the tangent line $y = 1+x$ lies below the curve $y = e^x$ as we look to both sides of the point of tangency $x = 0$. Suppose we wish to add a quadratic term to this linear approximation to make the resulting graph curve upwards a bit so that it is closer to the graph of $y = e^x$, at least near $x = 0$:

$$T_2(x) = 1 + x + cx^2.$$

What should the coefficient c be? Well, we expect $c > 0$ since we want the graph of $T_2(x)$ to curve upwards; but what value should we choose for c?

There are two clues to finding a reasonable value for c in what we have studied in this course so far:

1. c should have something to do with f, and in keeping with the way we constructed the linear approximation, we expect to use some piece of data about f at $x = 0$; and

2. we know that the *second* derivative, f'', tells us about the way $y = f(x)$ curves; that is, f'' tells us about the concavity of f.

So, with these two things in mind, let us ask of our approximant $T_2(x)$ something very basic: $T_2(x)$ should have the same second derivative at $x = 0$ as $f(x)$ does. (It already has the same function value and first derivative as $f(x)$ at $x = 0$, a fact you should verify if you don't see it right away.) Thus, we ask that

$$T_2''(0) = f''(0). \tag{1}$$

Now, $T_2''(x) = 2c$ and so $T_2''(0) = 2c$. Also, $f''(x) = e^x$, so $f''(0) = e^0 = 1$. Hence, substituting these into (1), we find that

$$2c = 1,$$

which gives us

$$c = \frac{1}{2}.$$

Hence, $T_2(x) = 1 + x + \frac{1}{2}x^2$ is a second degree polynomial that agrees with $f(x) = e^x$ by having $T_2(0) = f(0)$, $T_2'(0) = f'(0)$, and $T_2''(0) = f''(0)$. We call $T_2(x)$ the *second degree Taylor polynomial for e^x about $x = 0$*. Taylor polynomials generated by looking at data at $x = 0$ are called also *Maclaurin polynomials*.

There is nothing that says we need to stop the process of constructing a Taylor (or Maclaurin) polynomial after the quadratic term. For $f(x) = e^x$, for example, we know that we can continue to take derivatives of f at $x = 0$ as many times as we like (we say e^x is *infinitely differentiable* in this case), and, indeed, its kth derivative is

$$f^{(k)}(x) = e^x,$$

and so $f^{(k)}(0) = e^0 = 1$ for $k = 0, 1, 2, \ldots$.

So, if we construct an nth degree polynomial

$$T_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

as an approximation to $f(x) = e^x$ by requiring that $p^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, ..., n$, then we find that

$$e^x \approx T_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n.$$

You can derive Taylor's formula for the coefficients c_k by using the fact that

$$T_n^{(k)}(x) = k!\, c_k + \text{ terms of higher degree in } x$$

to show that

$$c_k = \frac{f^{(k)}(0)}{k!}. \tag{2}$$

Note that the $k!$ arises since $(x^k)' = kx^{k-1}$ and so taking k successive derivatives of x^k gives you $k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1 \equiv k!$.

Example: Let us construct the fifth degree Maclaurin polynomial for the function $f(x) = \sin x$. That is, we wish to find the coefficients of

$$T_5(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^5 + c_4 x^4 + c_5 x^5.$$

First, we need the derivatives

$$\frac{d^k}{dx^k} \sin x \Big|_{x=0}$$

for $k = 0, 1, 2, ..., 5$:

$$\begin{aligned} &k = 0: \quad \sin(0) = 0, && k = 3: -\cos(0) = -1, \\ &k = 1: \quad \cos(0) = 1, && k = 4: \quad \sin(0) = 0, \\ &k = 2: -\sin(0) = 0, && k = 5: \quad \cos(0) = 1. \end{aligned}$$

Using Taylor's formula (2) for the coefficients, we find that

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Note that because $\sin(0) = 0$ and every even order derivative of $\sin x$ is $\pm \sin x$, we have only odd powers appearing with non-zero coefficients in $T_5(x)$. This is not surprising since $\sin x$ is an *odd* function; that is, $\sin(-x) = -\sin x$.

In general, we wish to use information about a function f at points other than $x = 0$ to construct an approximating polynomial of degree n. If we look at a function around the point given by $x = a$, Taylor polynomials look like

$$T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n,$$

where

$$c_k = \frac{f^{(k)}(a)}{k!}. \tag{3}$$

This form of the polynomial may look a little strange at first since you are likely used to writing your polynomials in simple powers of x, but it is very useful to write this polynomial in this form. In particular, if we follow Taylor's program to construct the coefficients c_k by making the derivatives $T_n^{(k)}(a) = f^{(k)}(a)$, then the calculation becomes trivial since

$$T_n^{(k)}(x) = k!\, c_k + \text{higher order terms in powers of } (x - a),$$

so that plugging in $x = a$ makes all the high-order terms vanish.

Example: Suppose we are asked to find the Taylor polynomial of degree 5 for $\sin x$ about $x = \frac{\pi}{2}$. This time, we lose the symmetry about the origin that gave us the expectation that we would only see odd terms. The derivatives at $x = \frac{\pi}{2}$ are

$$\begin{aligned} &k = 0: \quad \sin(\tfrac{\pi}{2}) = 1, && k = 3: -\cos(\tfrac{\pi}{2}) = 0, \\ &k = 1: \quad \cos(\tfrac{\pi}{2}) = 0, && k = 4: \quad \sin(\tfrac{\pi}{2}) = 1, \\ &k = 2: -\sin(\tfrac{\pi}{2}) = -1, && k = 5: \quad \cos(\tfrac{\pi}{2}) = 0. \end{aligned}$$

In fact, we are only left with even-order terms and the required polynomial has no x^5 term:

$$T_5(x) = 1 - \frac{1}{2!}(x - \tfrac{\pi}{2})^2 + \frac{1}{4!}(x - \tfrac{\pi}{2})^4.$$

Many of the basic functions you know have useful Maclaurin polynomial approximations. If you wish an approximation of degree n, then

1. $e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$;
2. $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, where $2k+1$ is the greatest *odd* integer less than or equal to n;
3. $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!}$, where $2k$ is the greatest *even* integer less than or equal to n;
4. $\ln(1-x) \approx -x - \frac{x^2}{2} - \cdots - \frac{x^n}{n}$;
5. $\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^k x^{2k+1}}{2k+1}$, where $2k+1$ is the greatest odd integer less than or equal to n;
6. $\frac{1}{1-x} \approx 1 + x + x^2 + \cdots + x^n$.

EXERCISES:

1. Find the third degree Maclaurin polynomial for $f(x) = \sqrt{1+x}$.
2. Find the Taylor polynomials of degree 3 for $f(x) = 5x^2 - 3x + 2$ (a) about $x = -1$ and (b) about $x = 2$. What do you notice about them? If you expand each of these polynomials and collect powers of x, what do you notice?
3. If $f(x) = (1 + e^x)^2$, show that $f^{(k)}(0) = 2 + 2^k$ for any k. Write the Maclaurin polynomial of degree n for this function.
4. Find the Maclaurin polynomial of degree 5 for $f(x) = \tan x$.
5. Find the Maclaurin polynomial of degree 3 for $f(x) = e^{\sin x}$.

2 Taylor's Formula with Remainder

We constructed the Taylor polynomials hoping to approximate functions f by using information about the given function f at exactly one point $x = a$. **How well does the Taylor polynomial of degree n approximate the function f?**

One way of looking at this question is to ask for each value x, what is the difference between $f(x)$ and $T_n(x)$? If we call this difference the *remainder*, $R_n(x)$, we can write

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x). \qquad (4)$$

The first thing we notice if we look at (4) is that by taking this as the definition of $R_n(x)$, Taylor's formula (the rest of the right-hand side of (4)) is *automatically* correct. (This might take you a little thought to appreciate.) Of course, we would like to be able to deal with this remainder, $R_n(x)$, quantitatively. It turns out that we can use the Mean Value Theorem to find an expression for this remainder. The proof of this formula is a bit of a diversion from where we wish to go, so we will state the result without proof.

THE LAGRANGE REMAINDER FORMULA: Suppose that f has derivatives of at least order $n+1$ on some interval $[b,d]$. Then if x and a are any two numbers in (b,d), the remainder $R_n(x)$ in Taylor's formula can be written as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \tag{5}$$

where c is some number between x and a.

(Remark: The $n=0$ case is the Mean Value Theorem itself.)

First, note that c depends on both x and a. Now, if we could actually find this number c, we could know the remainder *exactly* for any given value of x. However, if you were to look at the proof of this formula, you would see that this number c comes into the formula because of the Mean Value Theorem. The Mean Value Theorem is very powerful, but all it tells us is that such a c exists, and not what its exact value is. Hence, we must figure out a way to use this Remainder Formula given our limited knowledge of c.

One approach is to ask ourselves: **What is the *worst* error we could make in approximating $f(x)$ using a Taylor polynomial of degree n about $x=a$?**

To answer this question, we will focus our attention on $|R_n(x)|$, the absolute value of the remainder. If we look at (5), we notice that we know everything except $f^{(n+1)}(c)$, and so, if our goal is to find a bound on the magnitude of the error, then we will need to find a bound on $|f^{(n+1)}(t)|$ that works for all values of t in the interval containing x and a. That is, we seek a positive number M such that

$$|f^{(n+1)}(t)| \leq M.$$

If we can find such an M, then we are able to bound the remainder, knowing x and a, as

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}. \tag{6}$$

EXAMPLE: Suppose we wish to compute $\sqrt{10}$ using a Taylor polynomial of degree $n=1$ (the linear approximation) for $a=9$ and give an estimate on the size of the error $|R_1(10)|$. First, we note that Taylor's formula for $f(x) = \sqrt{x}$ at $a = 9$ is given by

$$f(x) = f(9) + f'(9)(x-9) + R_1(x),$$

and so

$$\sqrt{x} = 3 + \frac{1}{6}(x-9) + R_1(x).$$

Thus, $\sqrt{10} \approx 3\frac{1}{6}$.

We now estimate $|R_1(10)|$. We first find M so that $|f''(t)| \le M$ for all t in $[9, 10]$. Now,

$$|f''(t)| = \left|\frac{-1}{4t^{3/2}}\right| = \frac{1}{4t^{3/2}}.$$

So, we want to make this function as big as possible on the interval $[9, 10]$. As t gets larger, $1/4t^{3/2}$ gets smaller, so it is largest at the left-hand endpoint, at $t = 9$. Hence, any value of M such that

$$M \ge \frac{1}{4 \cdot 9^{3/2}} = \frac{1}{108}$$

will work. We might as well choose $M = 1/108$ (though if you don't have a calculator, choosing $M = 1/100$ would make the computations easier if you wished to use decimal notation) and substitute this into (6) with $a = 9$ and $x = 10$ to get

$$|R_1(10)| \le \frac{1/108}{2!}|10-9|^2 = \frac{1}{216}.$$

Hence, we know that $\sqrt{10} = 3\frac{1}{6} \pm \frac{1}{216}$.

In fact, we can make a slightly stronger statement by noticing that the value of the second derivative, $f''(t)$, is always *negative* for t in the interval $[9, 10]$ and so we know that this tangent line always lies above the curve $y = \sqrt{x}$, and hence we are *overestimating* the value of $\sqrt{10}$ by using this linear approximation. Thus,

$$3\frac{1}{6} - \frac{1}{216} \le \sqrt{10} \le 3\frac{1}{6}.$$

Example: We approximate $\sin(0.5)$ by using a Maclaurin polynomial of degree 3. Recall that

$$\sin x = x - \frac{x^3}{3!} + R_3(x),$$

so

$$\sin(0.5) \approx 0.5 - \frac{0.5^3}{3!} = \frac{1}{2} - \frac{1}{48} = \frac{23}{48}.$$

To estimate the error in this approximation, we look for $M > 0$ such that

$$\left|\frac{d^4}{dt^4}\sin(t)\right| = |\sin(t)| \le M$$

for t in $[0, 0.5]$. The easiest choice for M is 1 since we know that $\sin(t)$ never gets larger than 1. However, we can do a bit better since we know that the tangent to $\sin(t)$ at $t = 0$ is $y = t$, which lies above the graph of $\sin(t)$ on $[0, 0.5]$. Thus, if we choose $M \geq 0.5$ we will get an appropriate bound. In this case,

$$|R_3(0.5)| \leq \frac{0.5}{4!}|0.5|^4 = \frac{1}{2 \cdot 24 \cdot 16} = \frac{1}{768}.$$

EXERCISES:

1. Find the second degree Taylor polynomial about $a = 10$ for $f(x) = 1/x$ and use it to compute $1/10.05$ to as many decimal places as is justified by this approximation.

2. What degree Maclaurin polynomial do you need to approximate $\cos(0.25)$ to 5 decimal places of accuracy?

3. Show that the approximation

   $$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{7!}$$

   gives the value of e to within an error of 8×10^{-5}.

4. ** If $f(x) = \sqrt{1 + x}$, show that $R_2(x)$, the remainder term associated to the second degree Maclaurin polynomial for $f(x)$, satisfies

   $$\frac{|x|^3}{16(1+x)^{5/2}} \leq |R_2(x)| \leq \frac{|x^3|}{16}$$

   for $x > 0$.

3 Taylor Series

We can use the Lagrange Remainder Formula to see how many functions can actually be represented completely by something called a *Taylor Series*. Stewart gives a fairly complete discussion of series, but we will make use of a simplified approach that is somewhat formal in nature since it suits our limited purposes.

We begin by considering something we will call a *power series* in $(x - a)$:

$$c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots, \tag{7}$$

where we choose the coefficients c_k to be real numbers. Of course, we are particularly interested in the case where we choose these coefficients to be those given by the Taylor formula, but there are also more general power series.

At first glance, the formula (7) looks very much like the polynomial formulae we have considered in the previous sections. However, the final $\cdots$ indicate that