SCIE 001 MATHEMATICS ASSIGNMENT 3 (Due 10:00 am Fri. Oct. 18, 2013)

There are two parts to this assignment. The first part is on WeBWorK and is due by 10:00 am on Fri. Oct. 18. The second part consists of the questions on this page. This assignment is due by 10:00 am on Fri. Oct. 18. For these questions, you are expected to provide full solutions with complete arguments and justifications. You will be graded on the correctness, clarity and elegance of your solutions. Your answers must be typeset or very neatly written. They must be stapled, with your name and student number at the top of each page.

1. Find all the strictly positive eigenvalues $\lambda = \lambda_n$ and their corresponding nonzero eigenfunctions $y = y_n(x)$, of

$$\begin{aligned} -y'' &= \lambda y, \quad 0 < x < L, \\ y'(0) &= 0, \qquad y(L) = 0, \end{aligned}$$

where L is a given positive constant. The eigenfunctions give the shapes of standing waves in a string of length L that is fixed at one end ($x = L$) and free to move vertically at the other end ($x = 0$; you could think of a horizontal string attached to a small ring that slides without friction on a vertical rod).

2. A differential equation can be made to "look nicer" with a change of variables, so that it is easier to solve. It can be shown that wavefunctions for the hydrogen atom have the form $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, where R, Θ and Φ are all functions of one variable and satisfy ordinary differential equations ($R(r)$ is the radial part, $\Theta(\theta)\Phi(\phi) = Y(\theta, \phi)$ is the angular part). The solutions of these ordinary differential equations give the orbital shapes you have seen in the chemistry lectures. It can be shown that the ordinary differential equation that Θ must satisfy is the rather nasty-looking

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[l(l+1) - \frac{m_l^2}{\sin^2\theta}\right]\Theta = 0, \tag{1}$$

where l and m_l are quantum numbers. Now make the change of variables

$$\Theta(\theta) = P(x), \quad \text{where} \quad x = \cos\theta.$$

(a) Use the Chain Rule to show that

$$\frac{d\Theta}{d\theta} = -\sin\theta \frac{dP}{dx} \qquad \text{and} \qquad \frac{d^2\Theta}{d\theta^2} = \sin^2\theta \frac{d^2P}{dx^2} - \cos\theta \frac{dP}{dx}.$$

(b) Use part (a) to help show that if $\Theta(\theta)$ is a solution of (1), then $P(x)$ must be a solution of

$$(1 - x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left[l(l+1) - \frac{m_l^2}{1 - x^2}\right]P = 0. \tag{2}$$

(c) It can be shown that (2) has polynomial solutions if $m_l = 0$. Verify by direct substitution that *i*) if $m_l = 0$ and $l = 1$ then $P(x) = x$ is a solution of (2) (and $\Theta(\theta) = \cos\theta$ is a solution of (1); solving $\Theta(\theta) = 0$ gives the nodal plane $\theta = \pi/2$ of a p_z orbital), *ii*) if $m_l = 0$ and $l = 2$ then $P(x) = 3x^2 - 1$ is a solution of (2) (and $\Theta(\theta) = 3\cos^2\theta - 1$ is a solution of (1); solving $\Theta(\theta) = 0$ give the nodal surfaces $\theta = \cos^{-1}(\pm 1/\sqrt{3})$ of a d_{z^2} orbital).

3. A tiny drop of oil, of radius 1.0×10^{-6} m, falling in air has a terminal velocity of 1.0×10^{-4} m/s. Suppose the oil drop has an electrical charge and moves downwards at 2.0×10^{-4} m/s due to an electrical field, and then the electrical field is switched off. Assume that the force due to air resistance is in the opposite direction to velocity and its magnitude is proportional to the speed of the oil drop. Also, assume the density of oil is 800 kg/m^3, and acceleration due to gravity is $g = 9.8$ m/s^2. Write down the relevant initial value problem for the *downwards* velocity $v(t)$ and solve it. How long after the electrical field is switched off does it take for the oil drop to slow to 1.1×10^{-4} m/s?