Homoclinic orbits

If \( p^0 \) is an equilibrium for
\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n,
\]
an orbit \( \Gamma = \{x^0(t)\}_{t \in \mathbb{R}} \) is \textbf{homoclinic} to \( p^0 \) if \( p^0 \notin \Gamma \) and \( \lim_{t \to \pm \infty} x^0(t) = p^0 \).

A homoclinic orbit to a hyperbolic equilibrium is structurally unstable: if there is a system with such a homoclinic orbit, then there are arbitrarily small perturbations of the system such that the phase portrait for the perturbed system in an open neighbourhood of \( \Gamma \cup \{p^0\} \) is not topologically equivalent to the original phase portrait. In particular, the perturbed system has hyperbolic equilibrium, but no homoclinic orbit.

Homoclinic (or saddle-loop) bifurcation for families of two-dimensional flows

Homoclinic orbits are often associated with other dynamics of interest. We summarize a bifurcation analysis near a homoclinic orbit for a family of autonomous ODEs in the plane. We consider a one-parameter family of two-dimensional ODEs
\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R},
\]
and assume there exist \( p^0_0 \in \mathbb{R}^2, \alpha_0 \in \mathbb{R} \) such that
\[
f(p^0_0, \alpha_0) = 0, \tag{SL.0.i}
\]
\[
A_0 = f_x(p^0_0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20}, \tag{SL.0.ii}
\]
for \( \alpha = \alpha_0 \), (4.1) has an orbit \( \Gamma = \{x^0(t)\} \) that is homoclinic to \( p^0_0 \). \tag{SL.0.iii}

Thus, for \( \alpha = \alpha_0 \), (4.1) has a hyperbolic saddle equilibrium \( p^0_0 \), \( \Gamma \) is a homoclinic orbit (or saddle loop), with \( \lim_{t \to \pm \infty} x^0(t) = p^0_0 \). Since a homoclinic orbit to a hyperbolic saddle equilibrium is structurally unstable, generically for \( \alpha \neq \alpha_0 \) there is no homoclinic orbit. We would like to determine other features of the dynamics for (4.1) when \( \alpha \neq \alpha_0 \).

By the usual arguments, the Implicit Function Theorem solves \( f(x, \alpha) = 0 \), to obtain a unique, locally defined, smooth solution \( x = p^0(\alpha) \), giving a smooth curve \( (p^0(\alpha), \alpha) \) of equilibria through the point \( (p^0_0, \alpha_0) \) in \( \mathbb{R}^2 \times \mathbb{R}^1 \). Since the eigenvalues of \( A_0 \) are simple, the matrix of the linearization \( A(\alpha) = f_x(p^0(\alpha), \alpha) \) has real eigenvalues \( \lambda_1(\alpha), \lambda_2(\alpha) \) that depend smoothly on \( \alpha \) near \( \alpha_0 \), with \( \lambda_j(\alpha_0) = \lambda_{j0} \), \( j = 1, 2 \), and thus by continuity \( \lambda_1(\alpha) < 0 < \lambda_2(\alpha) \) and \( p^0(\alpha) \) remains a hyperbolic saddle equilibrium, for all \( \alpha \) near \( \alpha_0 \).

As usual, we make a coordinate shift in \( \mathbb{R}^2 \)
\[
x = p^0(\alpha) + u, \tag{I}
\]
so that in \( (u_1, u_2) \) coordinates the equilibrium is \( (0, 0) \) for all \( \alpha \) near \( \alpha_0 \). Then we apply a family of linear coordinate changes in \( \mathbb{R}^2 \)
\[
u = T(\alpha)v, \tag{II}
\]
so that in \((v_1, v_2)\) coordinates, the linearization at the equilibrium \((0, 0)\) is diagonal (the real normal form), thus the local stable and unstable manifolds of the origin are tangent to the coordinate axes. Further smoothly invertible global nonlinear coordinate changes

\[ y = H(v, \alpha) \]

in \(\mathbb{R}^2\), for all \(\alpha\) near \(\alpha_0\), make the local stable and unstable manifolds coincide with the \(y_1\)- and \(y_2\)-coordinate axes along open segments containing the origin, and we obtain the family (4.1) expressed in new coordinates as

\[
\begin{align*}
\dot{y}_1 &= \lambda_1(\alpha)y_1 + \hat{h}_1(y_1, y_2, \alpha), \\
\dot{y}_2 &= \lambda_2(\alpha)y_2 + \hat{h}_2(y_1, y_2, \alpha),
\end{align*}
\]

(4.2)

for all \(\alpha\) near \(\alpha_0\), where

\[ \hat{h}_j(y_1, y_2, \alpha) = O(\|y_1, y_2\|^2), \quad j = 1, 2, \]

and also

\[ \hat{h}_1(0, y_2, \alpha) = 0, \quad \hat{h}_2(y_1, 0, \alpha) = 0, \]

for all \((y_1, y_2, \alpha)\) in an open neighbourhood of \((0, 0, \alpha_0)\). In these coordinates, for all \(\alpha\) near \(\alpha_0\), open segments of the coordinate axes \(\{y_1 = 0\}\) and \(\{y_2 = 0\}\) are locally invariant near \((0, 0)\).

Now we can begin our bifurcation analysis. We fix \(\varepsilon > 0\) sufficiently small, and choose a cross-section

\[ \Sigma = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = \varepsilon, |y_2| < \varepsilon\} \]

and consider the family of Poincaré maps

\[ P(\cdot, \alpha) : \Sigma^+ \to \Sigma, \quad \alpha \in \mathbb{R}^1, \]

where \(\Sigma^+\) is the portion of \(\Sigma\) with \(y_2 > 0\). Defining another cross-section

\[ \Pi = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1| < \varepsilon, y_2 = \varepsilon\}, \]

we express \(P\) as the composition of two parts \(P = Q \circ \Delta\), where the first part

\[ \Delta(\cdot, \alpha) : \Sigma^+ \to \Pi, \quad (\varepsilon, \eta_0) \mapsto (\xi, \varepsilon) \]

is the result of the flow near the origin where the system is approximately linear, and the second part

\[ Q(\cdot, \alpha) : \Pi \to \Sigma, \quad (\xi, \varepsilon) \mapsto (\varepsilon, \eta_1) \]

is the result of the flow near the homoclinic orbit \(\Gamma\) for \(\alpha\) near \(\alpha_0\).

An expression for \(\Delta\) is obtained by approximating (4.2) near the origin by its linearization, to obtain

\[ \xi = \Delta(\eta_0, \alpha) = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_1(\alpha)/\lambda_2(\alpha)} + \cdots = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_{10}/\lambda_{20}} + \cdots, \quad \eta_0 > 0 \]

(note that the exponents \(\lambda_1(\alpha)/\lambda_2(\alpha), \lambda_{10}/\lambda_{20}\) are negative).

For the flow near \(\Gamma\) for \(\alpha\) near \(\alpha_0\), we approximate \(Q\) by its Taylor polynomial at first order. We identify

\[ \beta(\alpha) = Q(0, \alpha) \]
as the split function that measures the signed distance from the stable manifold $W^s(p^0(\alpha))$ to the unstable manifold $W^u(p^0(\alpha))$, along $\Sigma$. Thus, assumption (SL.0.iii) is equivalent to

$$\beta(\alpha_0) = 0. \quad \text{(SL.0.iii')}$$

We further assume

$$a = \beta'(\alpha_0) \neq 0, \quad \text{(SL.1)}$$

and to leading order we have

$$\beta(\alpha) = a(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2).$$

Then

$$\eta_1 = Q(\xi, \alpha) = a(\alpha - \alpha_0) + \tilde{b}\xi + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||\xi| + |\xi|^2),$$

where $\tilde{b} > 0$ because $Q$ is a local diffeomorphism at $\xi = 0$ and orbits in $\mathbb{R}^2$ cannot cross, in particular for $\alpha = \alpha_0$.

Composing the two parts $Q$ and $\Delta$, we obtain the leading-order terms

$$\eta_1 = P(\eta_0, \alpha) = a(\alpha - \alpha_0) + b(\eta_0)^{-\lambda_{10}/\lambda_{20}} + \cdots, \quad \eta_0 > 0, \quad \text{(*)}$$

where $a \neq 0$ and $b > 0$ (note that the exponent $-\lambda_{10}/\lambda_{20}$ is positive). Finally, we assume that the trace of the matrix of the linearization at the saddle equilibrium does not vanish at the critical parameter value, i.e.

$$\sigma_0 = \text{div } f(p^0_0, \alpha_0) = \lambda_{10} + \lambda_{20} \neq 0, \quad \text{(SL.2)}$$

so that the positive exponent $-\lambda_{10}/\lambda_{20} \neq 1$. The transversality condition (SL.1) and nondegeneracy condition (SL.2) are generically satisfied. Theorem 4.1 (below) states that for sufficiently small $\varepsilon > 0$, under our assumptions the leading-order terms (*) are sufficient to determine the dynamics of the family of Poincaré maps $\eta \mapsto P(\eta, \alpha)$, up to local topological equivalence.

**Theorem 4.1. (Andronov & Leontovich)** If $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ is $C^2$ in an open set containing $\Gamma \cup \{p^0_0\}$ and satisfies the five conditions (SL.0.i)–(SL.2), then (4.1) has a family of Poincaré maps that is locally topologically equivalent to

$$\eta \mapsto \beta + b\eta^{-\lambda_{10}/\lambda_{20}}, \quad \eta > 0, \quad \text{at } (0,0),$$

where $b$ is a positive constant. In particular, for all $\alpha$ sufficiently near $\alpha_0$ there is an open neighbourhood $U$ of $\Gamma \cup \{p^0_0\}$ in $\mathbb{R}^2$, in which a unique limit cycle $L_\beta$ for (4.1) bifurcates from $\Gamma \cup \{p^0_0\}$ for $\alpha$ on only one side of $\alpha_0$. If $\sigma_0 < 0$, then $L_\beta$ is stable and exists only for $\beta > 0$, while if $\sigma_0 > 0$, then $L_\beta$ is unstable and exists only for $\beta < 0$.

The split function $\beta = a(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$ changes sign as $\alpha$ increases past $\alpha_0$. As $\alpha \to \alpha_0$ from the appropriate side of $\alpha_0$, $\beta \to 0$ and the limit cycle $L_\beta$ approaches $\Gamma \cup \{p^0_0\}$, and the period of $L_\beta$ approaches infinity.
Melnikov’s method

Melnikov’s method is a global perturbation method that allows us to prove the existence (or nonexistence) of homoclinic solutions, by perturbing from a case where a homoclinic solution is already known to exist.

For a two-dimensional autonomous Hamiltonian ODE (the “unperturbed” system) generated by a smooth Hamiltonian function $H(x)$,

\[ \dot{x} = f_0(x), \quad x \in \mathbb{R}^2, \]

where

\[ f_0 = \begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix}, \quad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = - \frac{\partial H}{\partial x_1}, \]

it is relatively easy to find homoclinic orbits. Let us assume the unperturbed system (4.3) has a hyperbolic saddle equilibrium $p_0$, and an orbit $\Gamma = \{x^0(t)\}$ homoclinic to $p_0$, $\lim_{t \to \pm\infty} x^0(t) = p_0$. (M.0)

Now we consider the family of “perturbed” two-dimensional nonautonomous periodic ODEs

\[ \dot{x} = f(x, t, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}^1, \]

where $\omega > 0$ is fixed, $\alpha$ is a parameter near 0, and $f$ is periodic in $t$ with period $T = 2\pi/\omega > 0$ ($f_1$ is periodic in $\omega t$ with period $2\pi$). Defining a new variable $\theta = \omega t$, we write the family of two-dimensional nonautonomous ODEs (4.4) as a family of three-dimensional autonomous ODEs

\[ \dot{x} = f_0(x) + \alpha f_1(x, \theta), \quad \dot{\theta} = \omega, \quad \bar{x} = (x, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 = X, \quad \alpha \in \mathbb{R}^1. \]

Defining a global cross-section for (4.5) for any $\alpha$,

\[ \Sigma = \{\bar{x} = (x, \theta) \in X : x \in \mathbb{R}^2, \theta = 0 \text{ (mod } 2\pi)\} = \mathbb{R}^2 \times \{0 \text{ (mod } 2\pi)\}, \]

we consider the family of two-dimensional Poincaré maps for (4.5), $P(\cdot, \alpha) : \Sigma \to \Sigma$.

For the unperturbed system $\alpha = 0$, the Poincaré map $P(\cdot, 0)$ is merely the time-$(2\pi/\omega)$ map for the flow of the unperturbed autonomous Hamiltonian system (4.3), so it has a hyperbolic saddle fixed point $(p_0^0, 0 \text{ (mod } 2\pi)) \in \Sigma$, whose one-dimensional stable and unstable manifolds $W_0^s, W_0^u$ coincide along the homoclinic orbit $\Gamma \times \{0 \text{ (mod } 2\pi)\}$ in $\Sigma$. Taking points in $\Sigma$ as initial values at $t = 0$ for the ODE (4.5), the hyperbolic saddle fixed point for $P(\cdot, 0)$ in $\Sigma$ generates a $(2\pi/\omega)$-periodic hyperbolic limit cycle $p_0^0(t) = (p_0^0, \omega t \text{ (mod } 2\pi))$ for (4.5) in $X$, whose two-dimensional stable and unstable manifolds $W_0^s$ and $W_0^u$ coincide along the homoclinic manifold $\bar{\Gamma} = \Gamma \times \mathbb{S}^1$ in $X$.

For the perturbed family $\alpha \neq 0$ sufficiently near 0, the hyperbolic saddle fixed point for the Poincaré map $P(\cdot, \alpha)$ persists as $(p^0(\alpha), 0 \text{ (mod } 2\pi)) \in \Sigma$ with $p^0(\alpha) = p_0^0 + O(|\alpha|)$, whose one-dimensional local stable and unstable manifolds $W_{\alpha,\text{loc}}^s, W_{\alpha,\text{loc}}^u$ persist and are $O(|\alpha|)$-close to their $\alpha = 0$ counterparts $W_{0,\text{loc}}^s, W_{0,\text{loc}}^u$ in $\Sigma$. Taking points in $\Sigma$ as initial values at $t = 0$ for the ODE (4.5), it follows that the hyperbolic limit cycle for (4.5) persists as $\bar{p}^0(t, \alpha) = (p^0(t, \alpha), \omega t \text{ (mod } 2\pi))$ in $X$, where $\bar{p}^0(t, \alpha) = \bar{p}_0^0(t) + O(|\alpha|)$. It also follows that the two-dimensional local stable and
unstable manifolds $\tilde{W}^s_{\alpha,loc}$ and $\tilde{W}^u_{\alpha,loc}$ of $\tilde{p}^0(t, \alpha)$ also persist, and also remain $O(|\alpha|)$-close in $X$ to their $\alpha = 0$ counterparts $\tilde{W}^s_{0,loc}$ and $\tilde{W}^u_{0,loc}$. But globally the homoclinic manifold $\Gamma$ generically is lost for $\alpha \neq 0$.

We define the two-dimensional vector field

$$f_0^+ = \left( \begin{array}{c} -f_{20} \\ f_{10} \end{array} \right),$$

which is orthogonal to the unperturbed two-dimensional (Hamiltonian) vector field $f_0$. We define

$$M(\theta, \alpha) = \frac{\|f_0^+(x^0(0))\|_\beta(\theta, \alpha),}$$

where $\beta(\theta, \alpha)$ is the split function, the signed “horizontal” distance from $\tilde{W}^s_{\alpha}$ to $\tilde{W}^u_{\alpha}$ measured along a cross-section. The $\alpha$-derivative of $M$ at $\alpha = 0$ can be shown (see the discussion below) to be given by a **Melnikov integral**

$$M_\alpha(\theta, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(x^0(t), t + \theta/\omega, 0) \rangle \, dt = \int_{-\infty}^{+\infty} \langle f_0^+(x^0(t)), f_1(x^0(t), \omega t + \theta) \rangle \, dt, \quad (4.6)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^2$, and $\eta(t) = f_0^+(x^0(t))$.

**Theorem 4.2. (Melnikov’s Method for Nonautonomous Periodic Perturbations)** If $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ are $C^2$ and satisfy conditions (M.0) and

$$M_\alpha(\theta_0, 0) = 0, \quad M_{\alpha\theta}(\theta_0, 0) \neq 0 \quad \text{for some } \theta_0 \in \mathbb{S}^1, \quad (M.1)$$

then for all $\alpha \neq 0$ sufficiently close to 0, the stable and unstable manifolds $\tilde{W}^s_{\alpha}$ and $\tilde{W}^u_{\alpha}$ for (4.5) have transversal intersections. Furthermore, if $M_\alpha(\theta, 0) \neq 0$ for all $\theta \in \mathbb{S}^1$, then for all $\alpha \neq 0$ sufficiently close to 0, there is some open neighbourhood of $\Gamma \cup \{\tilde{p}^0_0\}$ in $X$ in which $\tilde{W}^s_{\alpha}$ and $\tilde{W}^u_{\alpha}$ do not intersect.

If the perturbed two-dimensional system is autonomous and depends on another parameter $\gamma \in \mathbb{R}^1$,

$$\dot{x} = f(x, \gamma, \alpha) = f_0(x) + \alpha f_1(x, \gamma), \quad x \in \mathbb{R}^2, \quad \gamma \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \quad (4.7)$$

we get a similar result by considering the appropriate Melnikov integral

$$M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(x^0(t), \gamma, 0) \rangle \, dt = \int_{-\infty}^{+\infty} \langle f_0^+(x^0(t)), f_1(x^0(t), \gamma) \rangle \, dt. \quad (4.7)$$

**Theorem 4.3. (Melnikov’s Method for Autonomous Perturbations)** If $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ are $C^2$ and satisfy conditions (M.0) and

$$M_\alpha(\gamma_0, 0) = 0, \quad M_{\alpha\gamma}(\gamma_0, 0) \neq 0 \quad \text{for some } \gamma_0 \in \mathbb{R}^1, \quad (M.2)$$

then for all $\alpha \neq 0$ sufficiently close to 0, there is an open neighbourhood $U$ of $\Gamma \cup \{\tilde{p}^0_0\}$ in $\mathbb{R}^2$, in which (4.7) has a homoclinic orbit only for a unique $\gamma = \gamma(\alpha) = \gamma_0 + O(|\alpha|)$ and no homoclinic orbit for $\gamma \neq \gamma(\alpha)$. Furthermore, if $M_\alpha(\gamma, 0) \neq 0$ for all $\gamma$, then for all $\alpha \neq 0$ sufficiently close to 0, (4.7) has no homoclinic orbit in $U$.

The sign of $M_\alpha(\gamma, 0)$ indicates how the stable and unstable manifolds of the saddle equilibrium split.