Homoclinic orbits

If \( p^0 \) is an equilibrium for
\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n,
\]
an orbit \( \Gamma = \{ x^0(t) \}_{t \in \mathbb{R}} \) is **homoclinic** to \( p^0 \) if \( p^0 \not\in \Gamma \) and \( \lim_{t \to \pm\infty} x^0(t) = p^0 \).

A homoclinic orbit to a hyperbolic equilibrium is structurally unstable: if there is a system with such a homoclinic orbit, then there are arbitrarily small perturbations of the system such that the phase portrait for the perturbed system in an open neighbourhood of \( \Gamma \cup \{ p^0 \} \) is not topologically equivalent to the original phase portrait. In particular, the perturbed system has hyperbolic equilibrium, but no homoclinic orbit.

Saddle loop (or homoclinic) bifurcation for one-parameter families of two-dimensional vector fields

Homoclinic orbits are often associated with other dynamics of interest. We summarize a bifurcation analysis near a homoclinic orbit for a family of vector fields in the plane. We consider a one-parameter family
\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1,
\]
and assume there exist \( p^0_0, \alpha^0_0 \in \mathbb{R}^2, \alpha^0_0 \in \mathbb{R}^1 \) such that
\[
f(p^0_0, \alpha^0_0) = 0 \quad \text{(equilibrium)},
\]
\[
A_0 = f_x(p^0_0, \alpha^0_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20} \quad \text{(saddle)},
\]
for \( \alpha = \alpha^0_0 \), (4.1) has an orbit \( \Gamma = \{ x^0(t) \} \) that is homoclinic to \( p^0_0 \) \quad \text{(bifurcation)}.

Thus, for \( \alpha = \alpha^0_0 \), (4.1) has a hyperbolic saddle equilibrium \( p^0_0, \Gamma \) is a homoclinic orbit (or saddle loop), with \( \lim_{t \to \pm\infty} x^0(t) = p^0_0 \). Since a homoclinic orbit to a hyperbolic saddle equilibrium is structurally unstable, generically for \( \alpha \neq \alpha^0_0 \) there is no homoclinic orbit. We would like to determine other features of the dynamics for (4.1) when \( \alpha \neq \alpha^0_0 \).

By the usual arguments, the Implicit Function Theorem solves \( f(x, \alpha) = 0 \), to obtain a unique, locally defined, smooth solution \( x = p^0(\alpha) \), giving a smooth curve \( (p^0(\alpha), \alpha) \) of equilibria through the point \( (p^0_0, \alpha^0_0) \) in \( \mathbb{R}^2 \times \mathbb{R}^1 \). Since the eigenvalues of \( A_0 \) are simple, the matrix of the linearization \( A(\alpha) = f_x(p^0(\alpha), \alpha) \) has real eigenvalues \( \lambda_1(\alpha), \lambda_2(\alpha) \) that depend smoothly on \( \alpha \) near \( \alpha^0_0 \), with \( \lambda_j(\alpha^0_0) = \lambda_j^0, \; j = 1, 2 \), and thus by continuity \( \lambda_1(\alpha) < 0 < \lambda_2(\alpha) \) and \( p^0(\alpha) \) remains a hyperbolic saddle equilibrium, for all \( \alpha \) near \( \alpha^0_0 \).

As usual, we make a coordinate shift in \( \mathbb{R}^2 \)
\[
x = p^0(\alpha) + u, \quad u \in \mathbb{R}^2,
\]
so that in \((u_1, u_2)\)-coordinates the equilibrium is \((0, 0)\) for all \( \alpha \) near \( \alpha^0_0 \). Then we apply a family of linear coordinate changes
\[
u = T(\alpha) v, \quad v \in \mathbb{R}^2,
\]
so that in \((v_1, v_2)\)-coordinates, the linearization at the equilibrium \((0, 0)\) is diagonal (the real normal form), thus the local stable and unstable manifolds of the origin are tangent to the coordinate axes. A family of smoothly invertible global nonlinear coordinate changes
\[ v = H(y, \alpha), \quad y \in \mathbb{R}^2, \]
for all \(\alpha\) near \(\alpha_0\), make the local stable and unstable manifolds \textit{coincide} with the \(y_1\)- and \(y_2\)-coordinate axes along open segments containing the origin in \((y_1, y_2)\)-coordinates, and we obtain the family (4.1) expressed in new coordinates as
\[
\begin{align*}
\dot{y}_1 &= \lambda_1(\alpha)y_1 + h_1(y_1, y_2, \alpha), \\
\dot{y}_2 &= \lambda_2(\alpha)y_2 + h_2(y_1, y_2, \alpha), \\
(\alpha, \eta) &\in \mathbb{R}^1, \\
(\dot{y}_1, \dot{y}_2) &\in \mathbb{R}^2,
\end{align*}
\]
for all \(\alpha\) near \(\alpha_0\), where
\[
h_j(y_1, y_2, \alpha) = O(\|y_1, y_2\|^2), \quad j = 1, 2,
\]
and also
\[
h_1(0, y_2, \alpha) = 0, \quad h_2(y_1, 0, \alpha) = 0,
\]
for all \((y_1, y_2, \alpha)\) in an open neighbourhood of \((0, 0, \alpha_0)\).

Now we begin our bifurcation analysis. We fix \(\varepsilon > 0\) sufficiently small, and choose a cross-section
\[
\Sigma = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = \varepsilon, |y_2| < \varepsilon\}
\]
and consider the family of Poincaré maps
\[
P(\cdot, \alpha) : \Sigma^+ \to \Sigma, \quad \alpha \in \mathbb{R}^1,
\]
where \(\Sigma^+\) is the portion of \(\Sigma\) with \(y_2 > 0\). Defining another cross-section
\[
\Pi = \{(y_1, y_2) \in \mathbb{R}^2 : |y_1| < \varepsilon, y_2 = \varepsilon\},
\]
we express \(P\) as the composition of two parts \(P = Q \circ \Delta\), where the first part
\[
\Delta(\cdot, \alpha) : \Sigma^+ \to \Pi, \quad (\varepsilon, \eta_0) \mapsto (\xi^*, \varepsilon)
\]
is the result of the flow near the origin where the system is approximately linear, and the second part
\[
Q(\cdot, \alpha) : \Pi \to \Sigma, \quad (\xi, \varepsilon) \mapsto (\varepsilon, \eta_1)
\]
is the result of the flow near the homoclinic orbit \(\Gamma\) for \(\alpha\) near \(\alpha_0\).

An expression for \(\Delta\) is obtained by approximating (4.2) near the origin by its linearization, to obtain
\[
\xi^* = \Delta(\eta_0, \alpha) = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_1(\alpha)/\lambda_2(\alpha)} + \cdots = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_{10}/\lambda_{20}} + \cdots, \quad \eta_0 > 0
\]
(note that the exponents \(\lambda_1(\alpha)/\lambda_2(\alpha), \lambda_{10}/\lambda_{20}\) are negative).

For \(y\) near \(\Gamma\), \(\alpha\) near \(\alpha_0\), we approximate \(Q\) by its Taylor polynomial at first order. We identify
\[
\beta(\alpha) = Q(0, \alpha)
\]
as the split function that measures the signed distance from the stable manifold $W^s(0)$ to the unstable manifold $W^u(0)$, along $\Sigma$. Thus, assumption (SL.0.iii) is equivalent to

$$\beta(\alpha_0) = 0 \quad (\text{bifurcation}).$$

(SL.0.iii')

We further assume

$$a = \beta'(\alpha_0) \neq 0 \quad (\text{transversality}),$$

(SL.1)

and to leading order we have

$$\beta(\alpha) = a(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2).$$

Then

$$\eta_1 = Q(\xi, \alpha) = a(\alpha - \alpha_0) + \bar{b} \xi + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||\xi| + |\xi|^2),$$

where $\bar{b} > 0$ because $Q$ is a local diffeomorphism at $\xi = 0$ and orbits in $\mathbb{R}^2$ cannot cross, in particular for $\alpha = \alpha_0$.

Composing the two parts $Q$ and $\Delta$, we obtain the leading-order terms

$$\eta_1 = P(\eta_0, \alpha) = a(\alpha - \alpha_0) + b(\eta_0)^{-\lambda_1/\lambda_2} + \cdots, \quad \eta_0 > 0,$$

(*)

where $a \neq 0$ and $b > 0$ (note that the exponent $-\lambda_1/\lambda_2$ is positive). Finally, we assume that the trace of the matrix of the linearization at the saddle equilibrium does not vanish at the critical parameter value, i.e.

$$\sigma_0 = \text{div}\, f(p_0^0, \alpha_0) = \lambda_1 + \lambda_2 \neq 0 \quad (\text{nondegeneracy}),$$

(SL.2)

so that the exponent $-\lambda_1/\lambda_2 \neq 1$. The transversality condition (SL.1) and nondegeneracy condition (SL.2) are generically satisfied. Theorem 4.1 (below) states that for sufficiently small $\varepsilon > 0$, under our assumptions the leading-order terms (*) are sufficient to determine the dynamics of the family of Poincaré maps $\eta \mapsto P(\eta, \alpha)$, up to local topological equivalence.

**Theorem 4.1. (Andronov & Leontovich)** If $f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ is $C^2$ in an open set containing $\Gamma \cup \{p_0^0\}$ and satisfies the five conditions (SL.0.i)–(SL.2), then (4.1) has a family of Poincaré maps that is locally topologically equivalent to

$$\eta \mapsto \beta + b \eta^{-\lambda_1/\lambda_2}, \quad \eta > 0, \quad \text{at } (0,0),$$

where $b$ is a positive constant. In particular, for all $\alpha$ sufficiently near $\alpha_0$ there is an open neighbourhood $U$ of $\Gamma \cup \{p_0^0\}$ in $\mathbb{R}^2$, in which a unique limit cycle $L_\beta$ for (4.1) bifurcates from $\Gamma \cup \{p_0^0\}$ for $\alpha$ on only one side of $\alpha_0$. If $\sigma_0 < 0$, then $L_\beta$ is stable and exists only for $\beta > 0$, while if $\sigma_0 > 0$, then $L_\beta$ is unstable and exists only for $\beta < 0$.

The split function $\beta = a(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$ changes sign as $\alpha$ increases past $\alpha_0$. As $\alpha \to \alpha_0$ from the appropriate side of $\alpha_0$, $\beta \to 0$ and the limit cycle $L_\beta$ approaches $\Gamma \cup \{p_0^0\}$, and the period of $L_\beta$ approaches infinity.
Melnikov’s method

Melnikov’s method is a global perturbation method that allows us to prove the existence (or nonexistence) of homoclinic solutions, by perturbing from a case where a homoclinic solution is already known to exist.

For a two-dimensional autonomous Hamiltonian vector field (the “unperturbed” system) generated by a smooth Hamiltonian function $H(x)$,

$$\dot{x} = f_0(x), \quad x \in \mathbb{R}^2,$$

where

$$f_0 = \begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix}, \quad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},$$

it is relatively easy to find homoclinic orbits. Let us assume

the vector field (4.3) has a hyperbolic saddle equilibrium $p_0^0$, and an orbit $\Gamma = \{x^0(t)\}$ homoclinic to $p_0^0$, \( \lim_{t \to \pm \infty} x^0(t) = p_0^0 \).

(M.0)

Now we consider the family of “perturbed” two-dimensional nonautonomous periodic ordinary differential equations

$$\dot{x} = f(t, x, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1,$$

where $\omega > 0$ is fixed, $\alpha$ is a parameter near 0, and $f$ is periodic in $t$ with period $T = 2\pi/\omega > 0$ ($f_1$ is periodic in $\omega t$ with period $2\pi$). Defining a new variable $\theta = \omega t$, we write the family of two-dimensional nonautonomous differential equations (4.4) as a family of three-dimensional autonomous differential equations

$$\dot{x} = f_0(x) + \alpha f_1(x, \theta), \quad \dot{\theta} = \omega, \quad \tilde{x} = (x, \theta) \in \mathbb{R}^2 \times S^1 = X, \quad \alpha \in \mathbb{R}^1.$$

(4.5)

Defining a global cross-section for (4.5) for any $\alpha$,

$$\Sigma_0 = \{\tilde{x} = (x, \theta) \in X: x \in \mathbb{R}^2, \theta = 0 \pmod{2\pi}\} = \mathbb{R}^2 \times \{0 \pmod{2\pi}\},$$

we study the family of two-dimensional Poincaré maps for (4.5), $P(\cdot, \alpha): \Sigma_0 \to \Sigma_0$.

For the unperturbed $\alpha = 0$ system, the Poincaré map $P(\cdot, 0)$ is just the time-$(2\pi/\omega)$ map for the flow of the unperturbed autonomous Hamiltonian system (4.3), so it has a hyperbolic saddle fixed point $(p_0^0, 0 \pmod{2\pi}) \in \Sigma_0$, whose one-dimensional stable and unstable manifolds intersect along the smooth curve $\Gamma \times \{0 \pmod{2\pi}\}$ in $\Sigma_0$. Taking points in $\Sigma_0$ as initial values at $t = 0$ for (4.5)$_{\alpha=0}$, the hyperbolic saddle fixed point for $P(\cdot, 0)$ in $\Sigma_0$ generates a $(2\pi/\omega)$-periodic hyperbolic limit cycle $\tilde{p}_0^0(t) = (p_0^0, \omega t \pmod{2\pi})$ for (4.5)$_{\alpha=0}$ in $X$, whose two-dimensional stable and unstable manifolds $W^s_0$ and $W^u_0$ intersect along the smooth homoclinic manifold $\tilde{\Gamma} = \Gamma \times S^1$ in $X$.

For the perturbed system with $\alpha \neq 0$ sufficiently near 0, the hyperbolic saddle fixed point for the Poincaré map $P(\cdot, \alpha)$ persists as $(p_0^0(\alpha), 0 \pmod{2\pi}) \in \Sigma_0$ with $p_0^0(\alpha) = p_0^0 + O(|\alpha|)$, whose one-dimensional local stable and unstable manifolds persist in $\Sigma_0$ and are $O(|\alpha|)$-close to their $\alpha = 0$ counterparts. Taking points in $\Sigma_0$ as initial values at $t = 0$ for (4.5), it follows that the hyperbolic limit cycle for (4.5) persists as $\tilde{p}_0^0(t, \alpha) = (p_0^0(t, \alpha), \omega t \pmod{2\pi})$ in $X$, where $p_0^0(0, \alpha) = p_0^0(\alpha)$.
and \( p^0(t, \alpha) = \bar{p}^0(t) + O(|\alpha|) \). It also follows that the two-dimensional local stable and unstable manifolds \( \tilde{W}^s_{\alpha,loc} \) and \( \tilde{W}^u_{\alpha,loc} \) of \( p^0(t, \alpha) \) also persist in \( X \), and also remain \( O(|\alpha|) \)-close to their \( \alpha = 0 \) counterparts \( W^s_{\alpha,loc} \) and \( W^u_{\alpha,loc} \). But globally the smooth homoclinic manifold \( \tilde{\Gamma} \) generically is lost for \( \alpha \neq 0 \).

We define the two-dimensional vector field

\[
f_0^\perp = \begin{pmatrix} -f_{20} \\ f_{10} \end{pmatrix},
\]

which is orthogonal to the unperturbed two-dimensional (Hamiltonian) vector field \( f_0 \). We define

\[
M(\theta, \alpha) = \|f_0^\perp(x^0(0))\| \beta(\theta, \alpha),
\]

where \( \beta(\theta, \alpha) \) is the split function, the signed “horizontal” distance from \( \tilde{W}^s_{\alpha} \) to \( \tilde{W}^u_{\alpha} \) measured along a cross-section. The \( \alpha \)-derivative of \( M \) at \( \alpha = 0 \) can be shown (see the discussion below) to be given by a Melnikov integral

\[
M_\alpha(\theta, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(t + \theta/\omega, x^0(t), 0) \rangle \, dt = \int_{-\infty}^{+\infty} \langle f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta) \rangle \, dt, \tag{4.6}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^2 \), and \( \eta(t) = f_0^\perp(x^0(t)) \).

**Theorem 4.2.** (Melnikov’s Method for Nonautonomous Periodic Perturbations) If \( f_0 : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \) are \( C^2 \), with \( f_1 \) periodic in the last variable with period \( 2\pi \), and conditions (M.0) and

\[
M_\alpha(\theta, 0) = 0, \quad M_{\alpha\theta}(\theta, 0) \neq 0 \quad \text{for some } \theta_0 \in \mathbb{S}^1, \tag{M.1}
\]

are satisfied, then for all \( \alpha \neq 0 \) sufficiently close to 0, there is an open neighbourhood \( \tilde{U} \) of \( \tilde{\Gamma} \cup \{\bar{p}_0^0(t)\} \) in \( X \), in which the stable and unstable manifolds \( \tilde{W}^s_\alpha \) and \( \tilde{W}^u_\alpha \) for (4.5) have transversal intersections. Furthermore, if \( M_\alpha(\theta, 0) \neq 0 \) for all \( \theta \in \mathbb{S}^1 \), then for all \( \alpha \neq 0 \) sufficiently close to 0, \( \tilde{W}^s_\alpha \) and \( \tilde{W}^u_\alpha \) do not intersect in \( U \).

If the perturbed two-dimensional system is autonomous and depends on another parameter \( \gamma \in \mathbb{R}^1 \),

\[
\dot{x} = f(x, \gamma, \alpha) = f_0(x) + \alpha f_1(x, \gamma), \quad x \in \mathbb{R}^2, \quad \gamma \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1, \tag{4.7}
\]

we get a similar result by considering the appropriate Melnikov integral

\[
M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(x^0(t), \gamma, 0) \rangle \, dt = \int_{-\infty}^{+\infty} \langle f_0^\perp(x^0(t)), f_1(x^0(t), \gamma) \rangle \, dt. \tag{4.8}
\]

**Theorem 4.3.** (Melnikov’s Method for Autonomous Perturbations) If \( f_0 : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \) are \( C^2 \) and satisfy conditions (M.0) and

\[
M_\alpha(\gamma_0, 0) = 0, \quad M_{\alpha\gamma}(\gamma_0, 0) \neq 0 \quad \text{for some } \gamma_0 \in \mathbb{R}^1, \tag{M.2}
\]

then for all \( \alpha \neq 0 \) sufficiently close to 0, there is an open neighbourhood \( U \) of \( \Gamma \cup \{p_0^0\} \) in \( \mathbb{R}^2 \), in which (4.7) has a homoclinic orbit only for a unique \( \gamma = \hat{\gamma}(\alpha) = \gamma_0 + O(|\alpha|) \) and no homoclinic orbit for \( \gamma \neq \hat{\gamma}(\alpha) \). Furthermore, if \( M_\alpha(\gamma, 0) \neq 0 \) for all \( \gamma \), then for all \( \alpha \neq 0 \) sufficiently close to 0, (4.7) has no homoclinic orbit in \( U \).
The sign of $M_\alpha(\gamma,0)$ indicates how the stable and unstable manifolds of the saddle equilibrium split.

(Example 4.A was done in class.)

Discussion of proof of Theorem 4.2. We want to determine the relative positions of the two-dimensional global stable and unstable manifolds $\tilde{W}_s^\alpha$ and $\tilde{W}_u^\alpha$ of the limit cycle $\tilde{p}_0(t,\alpha)$ in $X$, in some fixed, open three-dimensional neighbourhood $U$, of $\tilde{\Gamma} \cup \{\tilde{p}_0^0(t)\}$ in $X$, for all $\alpha$ near 0. To do this, we use the vector field $f_0^\perp$ to define another, “vertical” cross-section for the family of vector fields (4.5),

$$\Pi = \{(x,\theta) \in X : x = x^0(0) + \beta f_0^\perp(x^0(0))/\|f_0^\perp(x^0(0))\|, \theta \in S^1, \beta \in \mathbb{R}^1 \text{ near } 0\}$$

for all $\alpha$ near 0. For $\alpha = 0$, $\Pi$ has orthogonal (and therefore transversal) intersection with the homoclinic manifold $\tilde{\Gamma} \subset \tilde{W}_0^u \cap \tilde{W}_0^s$ in $X$. Therefore for all $\alpha$ near 0, $\Pi$ still has transversal intersections with both $\tilde{W}_0^s$ and $\tilde{W}_0^u$, which we denote by the smooth curves $(x^s_0(\theta,\alpha), \theta \text{ (mod } 2\pi))$ and $(x^u_0(\theta,\alpha), \theta \text{ (mod } 2\pi))$ in $\Pi$, respectively. The split function $\beta(\theta,\alpha)$ is the signed “horizontal” (i.e. constant-$\theta$) distance between these curves in $\Pi$,

$$\beta(\theta,\alpha) = M(\theta,\alpha)/\|f_0^\perp(x^0(0))\|, \quad M(\theta,\alpha) = \langle f_0^\perp(x^0(0)), x^u_0(\theta,\alpha) - x^s_0(\theta,\alpha) \rangle.$$ 

Since the “Melnikov” function $M(\theta,\alpha)$ is a positive constant multiple of the split function, $\tilde{W}_0^s$ and $\tilde{W}_0^u$ intersect (along $\Pi$) if and only if

$$M(\theta,\alpha) = 0$$

for some $\theta \in S^1$. 