Homoclinic orbits

If \( p^0 \) is an equilibrium for
\[
\dot{x} = f(x), \quad x \in \mathbb{R}^n,
\]
an orbit \( \Gamma = \{x^0(t)\}_{t \in \mathbb{R}} \) is homoclinic to \( p^0 \) if \( p^0 \not\in \Gamma \) and \( \lim_{t \to +\infty} x^0(t) = p^0 = \lim_{t \to -\infty} x^0(t) \).

A homoclinic orbit to a hyperbolic equilibrium is structurally unstable: if there is a vector field with such a homoclinic orbit, then there are arbitrarily small perturbations of the vector field such that the perturbed vector field in an open neighbourhood of \( \Gamma \cup \{p^0\} \) is not topologically equivalent to the original vector field, in particular, the perturbed vector field has no homoclinic orbit.

The homoclinic bifurcation, for families of two-dimensional vector fields

Homoclinic orbits are often associated with other dynamics of interest. We summarize a bifurcation analysis near a homoclinic orbit for a family of vector fields in the plane. For the unperturbed system, this orbit is homoclinic to a saddle point, and sometimes is referred to as a “saddle-loop”. We consider a one-parameter family
\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1,
\]
and assume there exist \( p^0_0 \in \mathbb{R}^2, \alpha_0 \in \mathbb{R}^1 \) such that
\[
f(p^0_0, \alpha^0) = 0 \quad (equilibrium), \quad A_0 = f_x(p^0_0, \alpha_0) \text{ has eigenvalues } \lambda_{10} < 0 < \lambda_{20} \quad (saddle),
\]
\[ (4.1.0) \text{ for } \alpha = \alpha_0 \text{ has an orbit } \Gamma = \{x^0(t)\} \text{ that is homoclinic to } p^0_0 \quad (bifurcation). \quad (SL.0.iii) \]

Thus, for \( \alpha = \alpha_0 \), \( (4.1.0) \) has a hyperbolic saddle equilibrium \( p^0_0 \) and there is a homoclinic orbit \( \Gamma = \{x^0(t)\} \), with \( \lim_{t \to \pm\infty} x^0(t) = p^0_0 \). Generically, for \( \alpha \neq \alpha_0 \), \( \alpha \) near \( \alpha_0 \), there is no homoclinic orbit for \( (4.1.0) \) in an open neighbourhood of \( \Gamma \cup \{p^0_0\} \). We would like to determine other features of the dynamics for \( (4.1.0) \) when \( \alpha \neq \alpha_0 \).

By the usual arguments, the Implicit Function Theorem solves \( f(x, \alpha) = 0 \), to obtain a unique, locally defined, smooth solution \( x = p^0(\alpha) \), giving a smooth curve \( (p^0(\alpha), \alpha) \) of isolated equilibria through the point \((p^0_0, \alpha_0)\) in \( \mathbb{R}^2 \times \mathbb{R}^1 \). Since the eigenvalues of \( A_0 \) are simple, the matrix of the linearization \( A(\alpha) = f_x(p^0(\alpha), \alpha) \) has real eigenvalues \( \lambda_1(\alpha), \lambda_2(\alpha) \) that depend smoothly on \( \alpha \) near \( \alpha_0 \), with \( \lambda_j(\alpha_0) = \lambda_{j0}, j = 1, 2 \), and thus by continuity \( \lambda_1(\alpha) < 0 < \lambda_2(\alpha) \) and \( p^0(\alpha) \) remains a hyperbolic saddle equilibrium, for all \( \alpha \) sufficiently near \( \alpha_0 \).

We make a coordinate shift in \( \mathbb{R}^2 \)
\[
x = p^0(\alpha) + u, \quad (I)
\]
so that in \((u_1, u_2)\)-coordinates the equilibrium is \((0, 0)\) for all \( \alpha \) sufficiently near \( \alpha_0 \). The coordinate change \((I)\) transforms \( (4.1.0) \) into a family of the form
\[
\dot{u} = \hat{f}(u, \alpha), \quad u \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1, \quad (4.1.1)
\]
where \( \hat{f}(0, \alpha) = 0 \) for all \( \alpha \) sufficiently near \( \alpha_0 \), and \( \hat{f}(u, \alpha) = f(p^0(\alpha) + u, \alpha) \). Then we apply a family of linear coordinate changes

\[
u = T(\alpha) v,
\]

so that in \( v = (v_1, v_2) \)-coordinates, the linearization at the equilibrium \((0, 0)\) is diagonal (the real normal form), thus the local stable and unstable manifolds of the origin are tangent to the coordinate axes. The coordinate change (II) transforms (4.1.1) into a family of the form

\[
\begin{align*}
v_1 &= \lambda_1(\alpha) v_1 + g_1(v_1, v_2, \alpha), \\
v_2 &= \lambda_2(\alpha) v_2 + g_2(v_1, v_2, \alpha),
\end{align*}
\]

where \( g_j(v_1, v_2, \alpha) = O(\|\nu\|_2^2), \ j = 1, 2, \) for all \( \alpha \) sufficiently near \( \alpha_0 \).

Now a family of smoothly invertible global nonlinear coordinate changes

\[
v = H(y, \alpha)
\]

for all \( \alpha \) sufficiently near \( \alpha_0 \), make the local stable and unstable manifolds coincide with the coordinate axes along open segments containing the origin in the \( y = (y_1, y_2) \)-coordinates, and the family (4.1.2) is transformed into one of the form

\[
\begin{align*}
y_1 &= \lambda_1(\alpha) y_1 + h_1(y_1, y_2, \alpha), \\
y_2 &= \lambda_2(\alpha) y_2 + h_2(y_1, y_2, \alpha),
\end{align*}
\]

for all \( \alpha \) sufficiently near \( \alpha_0 \), where \( h_j(y_1, y_2, \alpha) = O(\|y_1, y_2\|^2), \ j = 1, 2, \) and now also

\[
h_1(0, y_2, \alpha) = 0, \quad h_2(y_1, 0, \alpha) = 0,
\]

for all \((y_1, y_2, \alpha)\) in an open neighbourhood of \((0, 0, \alpha_0)\).

Now we begin our bifurcation analysis. We fix \( \varepsilon > 0 \) sufficiently small, and choose a cross-section

\[
\Sigma = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = \varepsilon, -\varepsilon < y_2 < \varepsilon\}
\]

and consider the family of Poincaré maps

\[
P(\cdot, \alpha) : \Sigma^+ \to \Sigma, \quad \alpha \in \mathbb{R}^1,
\]

where \( \Sigma^+ \) is the portion of \( \Sigma \) with \( 0 < y_2 < \varepsilon \). Defining another cross-section

\[
\Pi = \{(y_1, y_2) \in \mathbb{R}^2 : -\varepsilon < y_1 < \varepsilon, y_2 = \varepsilon\},
\]

we express \( P \) as the composition of two parts \( P = Q \circ \Delta \), where the first part

\[
\Delta(\cdot, \alpha) : \Sigma^+ \to \Pi, \quad (\varepsilon, \xi_0) \mapsto (\eta^*, \varepsilon)
\]

is the result of the flow near the origin where the system is approximately linear, and the second part

\[
Q(\cdot, \alpha) : \Pi \to \Sigma, \quad (\eta^*, \varepsilon) \mapsto (\varepsilon, \xi_1)
\]

is the result of the flow in an open neighbourhood of the homoclinic orbit \( \Gamma \), for \( \alpha \) sufficiently near \( \alpha_0 \).
An expression for $\Delta$ is obtained by approximating (4.1.3) near the origin by its linearization, to obtain

$$\eta^* = \Delta(\xi_0, \alpha) = \varepsilon \left( \frac{\varepsilon}{\xi_0} \right)^{\lambda_1(\alpha)/\lambda_2(\alpha)} + \cdots, \quad \xi_0 > 0$$

(note that the exponent $\lambda_1(\alpha)/\lambda_2(\alpha)$ is negative), where $\cdots$ denotes the error made by approximating (4.1.3) by the linearization.

For $y(t)$ near $\Gamma$, $\alpha$ near $\alpha_0$, we approximate $Q$ by its Taylor polynomial at first order. We identify

$$\beta(\alpha) = Q(0, \alpha)$$

as the split function that measures the signed distance from the stable manifold $W^s(0)$ to the unstable manifold $W^u(0)$, along $\Sigma$. Thus, assumption (SL.0.iii) is equivalent to

$$\beta(\alpha_0) = 0 \quad \text{(bifurcation).}$$

We further assume

$$a = \beta'(\alpha_0) \neq 0 \quad \text{(transversality),}$$

so to leading order we have

$$\beta(\alpha) = a (\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2).$$

Then

$$\xi_1 = Q(\eta, \alpha) = a (\alpha - \alpha_0) + Q_\eta(0, \alpha_0) \eta + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0| |\eta| + |\eta|^2),$$

where $Q_\eta(0, \alpha_0) > 0$ because $Q(\cdot, \alpha_0)$ is a local diffeomorphism at $\eta = 0$ and orbits in $\mathbb{R}^2$ cannot cross.

Composing the two parts $Q$ and $\Delta$, we obtain the leading-order terms

$$\xi_1 = P(\xi_0, \alpha) = a (\alpha - \alpha_0) + b (\xi_0)^{-\lambda_{10}/\lambda_{20}} + \cdots, \quad \xi_0 > 0,$$

(*)

where $a \neq 0$ and $b > 0$ (note that the exponent $-\lambda_{10}/\lambda_{20}$ is positive). Finally, we assume that the trace of the matrix of the linearization at the saddle equilibrium does not vanish at the critical parameter value, i.e.

$$\sigma_0 = \text{div } f(p^0_0, \alpha_0) = \lambda_{10} + \lambda_{20} \neq 0 \quad \text{(nondegeneracy)},$$

(SL.2)

so that the exponent $-\lambda_{10}/\lambda_{20} \neq 1$. The transversality condition (SL.1) and nondegeneracy condition (SL.2) are generically satisfied. Theorem 4.1 (below) states that for all sufficiently small $\varepsilon > 0$, under our assumptions the leading-order terms (*) are sufficient to determine the dynamics of the family of Poincaré maps $\xi \mapsto P(\xi, \alpha)$, up to local topological equivalence.

**Theorem 4.1. (Andronov & Leontovich)** If $f : \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{R}^2$ is $C^2$ in an open set containing $\Gamma \cup \{p^0_0\}$ and satisfies the five conditions (SL.0.i)–(SL.2), then (4.1.0) has a family of Poincaré maps that is locally topologically equivalent to

$$\xi \mapsto \beta + b \xi^{-\lambda_{10}/\lambda_{20}}, \quad \xi > 0, \quad \text{at } (0,0),$$

where $b$ is a positive constant. In particular, for all $\alpha$ sufficiently near $\alpha_0$ there is an open neighbourhood $U$ of $\Gamma \cup \{p^0_0\}$ in $\mathbb{R}^2$, in which a unique limit cycle $L_\beta$ for (4.1.0) bifurcates from $\Gamma \cup \{p^0_0\}$ for $\alpha$ on only one side of $\alpha_0$. If $\sigma_0 < 0$, then $L_\beta$ exists only for $\beta > 0$ and is stable, while if $\sigma_0 > 0$, then $L_\beta$ exists only for $\beta < 0$ and is unstable.

The split function $\beta = a (\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)$ changes sign as $\alpha$ increases past $\alpha_0$. As $\alpha \rightarrow \alpha_0$ from the appropriate side of $\alpha_0$, $\beta \rightarrow 0$ and the limit cycle $L_\beta$ approaches $\Gamma \cup \{p^0_0\}$, and the period of $L_\beta$ approaches infinity.