Homoclinic orbits

If $p^0$ is an equilibrium for

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

an orbit $\Gamma = \{x^0(t)\}_{t \in \mathbb{R}}$ is **homoclinic** to $p^0$ if $p^0 \notin \Gamma$ and $\lim_{t \to \pm \infty} x^0(t) = p^0$.

A homoclinic orbit to a hyperbolic equilibrium is structurally unstable: if there is a system with such a homoclinic orbit, then there are arbitrarily small perturbations of the system such that the phase portrait for the perturbed system in an open neighbourhood of $\Gamma \cup \{p^0\}$ is not topologically equivalent to the original phase portrait. In particular, the perturbed system has hyperbolic equilibrium, but no homoclinic orbit.

Saddle loop (or homoclinic) bifurcation for one-parameter families of two-dimensional vector fields

Homoclinic orbits are often associated with other dynamics of interest. We summarize a bifurcation analysis near a homoclinic orbit for a family of vector fields in the plane. We consider a one-parameter family

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1, \quad (4.1)$$

and assume there exist $p^0_0 \in \mathbb{R}^2$, $\alpha_0 \in \mathbb{R}^1$ such that

$$f(p^0_0, \alpha_0) = 0 \quad (equilibrium), \quad (SL.0.i)$$

$$A_0 = f_x(p^0_0, \alpha_0)$$

has eigenvalues $\lambda_{10} < 0 < \lambda_{20} \quad (saddle), \quad (SL.0.ii)$$

for $\alpha = \alpha_0$, (4.1) has an orbit $\Gamma = \{x^0(t)\}$ that is homoclinic to $p^0_0$ \quad (bifurcation). \quad (SL.0.iii)$$

Thus, for $\alpha = \alpha_0$, (4.1) has a hyperbolic saddle equilibrium $p^0_0$, $\Gamma$ is a homoclinic orbit (or saddle loop), with $\lim_{t \to \pm \infty} x^0(t) = p^0_0$. Since a homoclinic orbit to a hyperbolic saddle equilibrium is structurally unstable, generically for $\alpha \neq \alpha_0$ there is no homoclinic orbit. We would like to determine other features of the dynamics for (4.1) when $\alpha \neq \alpha_0$.

By the usual arguments, the Implicit Function Theorem solves $f(x, \alpha) = 0$, to obtain a unique, locally defined, smooth solution $x = p^0(\alpha)$, giving a smooth curve $(p^0(\alpha), \alpha)$ of equilibria through the point $(p^0_0, \alpha_0)$ in $\mathbb{R}^2 \times \mathbb{R}^1$. Since the eigenvalues of $A_0$ are simple, the matrix of the linearization $A(\alpha) = f_x(p^0(\alpha), \alpha)$ has real eigenvalues $\lambda_1(\alpha)$, $\lambda_2(\alpha)$ that depend smoothly on $\alpha$ near $\alpha_0$, with $\lambda_j(\alpha_0) = \lambda_{j0}$, $j = 1, 2$, and thus by continuity $\lambda_1(\alpha) < 0 < \lambda_2(\alpha)$ and $p^0(\alpha)$ remains a hyperbolic saddle equilibrium, for all $\alpha$ near $\alpha_0$.

As usual, we make a coordinate shift in $\mathbb{R}^2$

$$x = p^0(\alpha) + u, \quad u \in \mathbb{R}^2, \quad (I)$$

so that in $(u_1, u_2)$-coordinates the equilibrium is $(0, 0)$ for all $\alpha$ near $\alpha_0$. Then we apply a family of linear coordinate changes

$$u = T(\alpha) v, \quad v \in \mathbb{R}^2, \quad (II)$$
so that in \((v_1, v_2)\)-coordinates, the linearization at the equilibrium \((0, 0)\) is diagonal (the real normal form), thus the local stable and unstable manifolds of the origin are tangent to the coordinate axes. A family of smoothly invertible global nonlinear coordinate changes 

\[ v = H(y, \alpha), \quad y \in \mathbb{R}^2, \]

for all \(\alpha\) near \(\alpha_0\), make the local stable and unstable manifolds coincide with the \(y_1\) and \(y_2\)-coordinate axes along open segments containing the origin in \((y_1, y_2)\)-coordinates, and we obtain the family (4.1) expressed in new coordinates as

\[
\begin{align*}
\dot{y}_1 &= \lambda_1(\alpha)y_1 + h_1(y_1, y_2, \alpha), \\
\dot{y}_2 &= \lambda_2(\alpha)y_2 + h_2(y_1, y_2, \alpha),
\end{align*}
\]

for all \(\alpha\) near \(\alpha_0\), where

\[ h_j(y_1, y_2, \alpha) = O(\| (y_1, y_2) \|^2), \quad j = 1, 2, \]

and also

\[ h_1(0, y_2, \alpha) = 0, \quad h_2(y_1, 0, \alpha) = 0, \]

for all \((y_1, y_2, \alpha)\) in an open neighbourhood of \((0, 0, \alpha_0)\).

Now we begin our bifurcation analysis. We fix \(\varepsilon > 0\) sufficiently small, and choose a cross-section

\[ \Sigma = \{ (y_1, y_2) \in \mathbb{R}^2 : y_1 = \varepsilon, |y_2| < \varepsilon \} \]

and consider the family of Poincaré maps

\[ P(\cdot, \alpha) : \Sigma^+ \to \Sigma, \quad \alpha \in \mathbb{R}^1, \]

where \(\Sigma^+\) is the portion of \(\Sigma\) with \(y_2 > 0\). Defining another cross-section

\[ \Pi = \{ (y_1, y_2) \in \mathbb{R}^2 : |y_1| < \varepsilon, y_2 = \varepsilon \}, \]

we express \(P\) as the composition of two parts \(P = Q \circ \Delta\), where the first part

\[ \Delta(\cdot, \alpha) : \Sigma^+ \to \Pi, \quad (\varepsilon, \eta_0) \mapsto (\xi^*, \varepsilon) \]

is the result of the flow near the origin where the system is approximately linear, and the second part

\[ Q(\cdot, \alpha) : \Pi \to \Sigma, \quad (\xi, \varepsilon) \mapsto (\varepsilon, \eta_1) \]

is the result of the flow near the homoclinic orbit \(\Gamma\) for \(\alpha\) near \(\alpha_0\).

An expression for \(\Delta\) is obtained by approximating (4.2) near the origin by its linearization, to obtain

\[ \xi^* = \Delta(\eta_0, \alpha) = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_1(\alpha)/\lambda_2(\alpha)} + \cdots = \varepsilon \left( \frac{\varepsilon}{\eta_0} \right)^{\lambda_{10}/\lambda_{20}} + \cdots, \quad \eta_0 > 0 \]

(note that the exponents \(\lambda_1(\alpha)/\lambda_2(\alpha)\), \(\lambda_{10}/\lambda_{20}\) are negative).

For \(y\) near \(\Gamma\), \(\alpha\) near \(\alpha_0\), we approximate \(Q\) by its Taylor polynomial at first order. We identify

\[ \beta(\alpha) = Q(0, \alpha) \]
as the **split function** that measures the signed distance from the stable manifold \(W^s(0)\) to the unstable manifold \(W^u(0)\), along \(\Sigma\). Thus, assumption (SL.0.iii) is equivalent to
\[
\beta(\alpha_0) = 0 \quad \text{(bifurcation)}. \quad \text{(SL.0.iii')}
\]

We further assume
\[
a = \beta'(\alpha_0) \neq 0 \quad \text{(transversality)}, \quad \text{(SL.1)}
\]
and to leading order we have
\[
\beta(\alpha) = a (\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2).
\]

Then
\[
\eta_1 = Q(\xi, \alpha) = a (\alpha - \alpha_0) + \tilde{b} \xi + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||\xi| + |\xi|^2),
\]
where \(\tilde{b} > 0\) because \(Q\) is a local diffeomorphism at \(\xi = 0\) and orbits in \(\mathbb{R}^2\) cannot cross, in particular for \(\alpha = \alpha_0\).

Composing the two parts \(Q\) and \(\Delta\), we obtain the leading-order terms
\[
\eta_1 = P(\eta_0, \alpha) = a (\alpha - \alpha_0) + b (\eta_0)^{-\lambda_{10}/\lambda_{20}} + \cdots, \quad \eta_0 > 0, \quad \text{(*)}
\]
where \(a \neq 0\) and \(b > 0\) (note that the exponent \(-\lambda_{10}/\lambda_{20}\) is positive). Finally, we assume that the trace of the matrix of the linearization at the saddle equilibrium does not vanish at the critical parameter value, i.e.
\[
\sigma_0 = \text{div} f(p_0^0, \alpha_0) = \lambda_{10} + \lambda_{20} \neq 0 \quad \text{(nondegeneracy)}, \quad \text{(SL.2)}
\]

so that the exponent \(-\lambda_{10}/\lambda_{20} \neq 1\). The transversality condition (SL.1) and nondegeneracy condition (SL.2) are generically satisfied. Theorem 4.1 (below) states that for sufficiently small \(\varepsilon > 0\), under our assumptions the leading-order terms (*) are sufficient to determine the dynamics of the family of Poincaré maps \(\eta \mapsto P(\eta, \alpha)\), up to local topological equivalence.

**Theorem 4.1.** *(Andronov & Leontovich)* If \(f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2\) is \(C^2\) in an open set containing \(\Gamma \cup \{p_0^0\}\) and satisfies the five conditions (SL.0.i)—(SL.2), then (4.1) has a family of Poincaré maps that is locally topologically equivalent to
\[
\eta \mapsto \beta + b \eta^{-\lambda_{10}/\lambda_{20}}, \quad \eta > 0, \quad \text{at } (0,0),
\]
where \(b\) is a positive constant. In particular, for all \(\alpha\) sufficiently near \(\alpha_0\) there is an open neighbourhood \(U\) of \(\Gamma \cup \{p_0^0\}\) in \(\mathbb{R}^2\), in which a unique limit cycle \(L_\beta\) for (4.1) bifurcates from \(\Gamma \cup \{p_0^0\}\) for \(\alpha\) on only one side of \(\alpha_0\). If \(\sigma_0 < 0\), then \(L_\beta\) is stable and exists only for \(\beta > 0\), while if \(\sigma_0 > 0\), then \(L_\beta\) is unstable and exists only for \(\beta < 0\).

The split function \(\beta = a (\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2)\) changes sign as \(\alpha\) increases past \(\alpha_0\). As \(\alpha \to \alpha_0\) from the appropriate side of \(\alpha_0\), \(\beta \to 0\) and the limit cycle \(L_\beta\) approaches \(\Gamma \cup \{p_0^0\}\), and the period of \(L_\beta\) approaches infinity.
Melnikov’s method

Melnikov’s method is a global perturbation method that allows us to prove the existence (or nonexistence) of homoclinic solutions, by perturbing from a case where a homoclinic solution is already known to exist.

For a two-dimensional autonomous Hamiltonian vector field (the “unperturbed” system) generated by a smooth Hamiltonian function \( H(x) \),
\[
\dot{x} = f_0(x), \quad x \in \mathbb{R}^2,
\]
where
\[
f_0 = \begin{pmatrix} f_{10} \\ f_{20} \end{pmatrix}, \quad f_{10} = \frac{\partial H}{\partial x_2}, \quad f_{20} = -\frac{\partial H}{\partial x_1},
\]
it is relatively easy to find homoclinic orbits. Let us assume

the vector field (4.3) has a hyperbolic saddle equilibrium \( p_0^0 \),

and an orbit \( \Gamma = \{ x^0(t) \} \) homoclinic to \( p_0^0 \), \( \lim_{t \to \pm \infty} x^0(t) = p_0^0 \). (M.0)

Now we consider the family of “perturbed” two-dimensional nonautonomous periodic ordinary differential equations
\[
\dot{x} = f(t, x, \alpha) = f_0(x) + \alpha f_1(x, \omega t), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1,
\]
where \( \omega > 0 \) is fixed, \( \alpha \) is a parameter near 0, and \( f \) is periodic in \( t \) with period \( T = 2\pi/\omega > 0 \) (\( f_1 \) is periodic in \( \omega t \) with period \( 2\pi \)). Defining a new variable \( \theta = \omega t \), we write the family of two-dimensional nonautonomous differential equations (4.4) as a family of three-dimensional autonomous differential equations
\[
\dot{x} = f_0(x) + \alpha f_1(x, \theta), \quad \dot{\theta} = \omega, \quad \tilde{x} = (x, \theta) \in \mathbb{R}^2 \times S^1 = X, \quad \alpha \in \mathbb{R}^1.
\]

Defining a global cross-section for (4.5) for any \( \alpha \),
\[
\Sigma_0 = \{ \tilde{x} = (x, \theta) \in X : x \in \mathbb{R}^2, \theta = 0 \pmod{2\pi} \} = \mathbb{R}^2 \times \{ 0 \pmod{2\pi} \},
\]
we study the family of two-dimensional Poincaré maps for (4.5), \( P(\cdot, \alpha) : \Sigma_0 \to \Sigma_0 \).

For the unperturbed \( \alpha = 0 \) system, the Poincaré map \( P(\cdot, 0) \) is just the time-(2\( \pi/\omega \))-map for the flow of the unperturbed autonomous Hamiltonian system (4.3), so it has a hyperbolic saddle fixed point \( \tilde{p}_0^0, 0 \pmod{2\pi} \) \( \in \Sigma_0 \), whose one-dimensional stable and unstable manifolds intersect along the smooth curve \( \Gamma \times \{ 0 \pmod{2\pi} \} \) in \( \Sigma_0 \). Taking points in \( \Sigma_0 \) as initial values at \( t = 0 \) for (4.5)\( \alpha=0 \), the hyperbolic saddle fixed point for \( P(\cdot, 0) \) in \( \Sigma_0 \) generates a \( (2\pi/\omega) \)-periodic hyperbolic limit cycle \( \tilde{p}_0^0(t) = \tilde{p}_0^0, \omega t \pmod{2\pi} \) for (4.5)\( \alpha=0 \) in \( X \), whose two-dimensional stable and unstable manifolds \( \tilde{W}_0^s \) and \( \tilde{W}_0^u \) intersect along the smooth homoclinic manifold \( \tilde{\Gamma} = \Gamma \times S^1 \) in \( X \).

For the perturbed system with \( \alpha \neq 0 \) sufficiently near 0, the hyperbolic saddle fixed point for the Poincaré map \( P(\cdot, \alpha) \) persists as \( \tilde{p}^0(\alpha), 0 \pmod{2\pi} \) \( \in \Sigma_0 \) with \( \tilde{p}^0(\alpha) = \tilde{p}_0^0 + O(|\alpha|) \), whose one-dimensional local stable and unstable manifolds persist in \( \Sigma_0 \) and are \( O(|\alpha|) \)-close to their \( \alpha = 0 \) counterparts. Taking points in \( \Sigma_0 \) as initial values at \( t = 0 \) for (4.5), it follows that the hyperbolic limit cycle for (4.5) persists as \( \tilde{p}^0(t, \alpha) = (p^0(t, \alpha), \omega t \pmod{2\pi}) \) in \( X \), where \( p^0(0, \alpha) = \tilde{p}^0(\alpha) \).
which is orthogonal to the unperturbed two-dimensional (Hamiltonian) vector field $f$ for counterparts $\tilde{\alpha}$ manifolds $\tilde{\alpha}^\alpha$ across a section. The $\beta$ and $\tilde{\alpha}$ given by a Melnikov integral where

$$M(\theta, \alpha) = \frac{\lVert f_0^\perp(x^0(0)) \rVert}{\beta(\theta, \alpha)},$$

where $\beta(\theta, \alpha)$ is the split function, the signed “horizontal” distance from $\tilde{W}_\alpha^s$ to $\tilde{W}_\alpha^u$ measured along a cross-section. The $\alpha$-derivative of $M$ at $\alpha = 0$ can be shown (see the discussion below) to be given by a Melnikov integral

$$M_\alpha(\theta, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(t + \theta/\omega, x^0(0)) \rangle dt = \int_{-\infty}^{+\infty} \langle f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta) \rangle dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^2$, and $\eta(t) = f_0^\perp(x^0(t))$.

**Theorem 4.2.** (Melnikov’s Method for Nonautonomous Periodic Perturbations) If $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ are $C^2$, with $f_1$ periodic in the last variable with period $2\pi$, and conditions (M.0) and

$$M_\alpha(\theta_0, 0) = 0, \quad M_{\alpha\theta}(\theta_0, 0) \neq 0 \quad \text{for some } \theta_0 \in \mathbb{S}^1,$$

are satisfied, then for all $\alpha \neq 0$ sufficiently close to 0, there is an open neighbourhood $\tilde{U}$ of $\tilde{\Gamma} \cup \{\tilde{p}_0^\alpha(t)\}$ in $\mathbb{X}$, in which the stable and unstable manifolds $\tilde{W}_\alpha^s$ and $\tilde{W}_\alpha^u$ for (4.5) have transversal intersections. Furthermore, if $M_\alpha(\theta, 0) \neq 0$ for all $\theta \in \mathbb{S}^1$, then for all $\alpha \neq 0$ sufficiently close to 0, $\tilde{W}_\alpha^s$ and $\tilde{W}_\alpha^u$ do not intersect in $\tilde{U}$.

If the perturbed two-dimensional system is autonomous and depends on another parameter $\gamma \in \mathbb{R}^1$,

$$\dot{x} = f(x, \gamma, \alpha) = f_0(x) + \alpha f_1(x, \gamma), \quad x \in \mathbb{R}^2, \quad \gamma \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,$$

we get a similar result by considering the appropriate Melnikov integral

$$M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(x^0(t), \gamma, 0) \rangle dt = \int_{-\infty}^{+\infty} \langle f_0^\perp(x^0(t)), f_1(x^0(t), \gamma) \rangle dt.$$

**Theorem 4.3.** (Melnikov’s Method for Autonomous Perturbations) If $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_1 : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ are $C^2$ and satisfy conditions (M.0) and

$$M_\alpha(\gamma_0, 0) = 0, \quad M_{\alpha\gamma}(\gamma_0, 0) \neq 0 \quad \text{for some } \gamma_0 \in \mathbb{R}^1,$$

then for all $\alpha \neq 0$ sufficiently close to 0, there is an open neighbourhood $U$ of $\Gamma \cup \{p_0^\alpha\}$ in $\mathbb{R}^2$, in which (4.7) has a homoclinic orbit only for a unique $\gamma = \hat{\gamma}(\alpha) = \gamma_0 + O(|\alpha|)$ and no homoclinic orbit for $\gamma \neq \hat{\gamma}(\alpha)$. Furthermore, if $M_\alpha(\gamma, 0) \neq 0$ for all $\gamma$, then for all $\alpha \neq 0$ sufficiently close to 0, (4.7) has no homoclinic orbit in $U$. 

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The sign of $M_\alpha(\gamma,0)$ indicates how the stable and unstable manifolds of the saddle equilibrium split.

(Example 4.A was done in class.)

**Discussion of proof of Theorem 4.2.** We want to determine the relative positions of the two-dimensional global stable and unstable manifolds $\bar{W}_s^\alpha$ and $\bar{W}_u^\alpha$ of the limit cycle $\bar{p}^0(t,\alpha)$ in $X$, in some fixed, open three-dimensional neighbourhood $U$, of $\bar{\Gamma} \cup \{\bar{p}_0^0(t)\}$ in $X$, for all $\alpha$ near 0. To do this, we use the vector field $f_0^\perp$ to define another, “vertical” cross-section for the family of vector fields (4.5),

$$\Pi = \{(x,\theta) \in X : x = x^0(0) + \beta f_0^\perp(x^0(0))/\|f_0^\perp(x^0(0))\|, \theta \in S^1, \beta \in \mathbb{R}^1 \text{ near } 0\}$$

for all $\alpha$ near 0. For $\alpha = 0$, $\Pi$ has orthogonal (and therefore transversal) intersection with the homoclinic manifold $\bar{\Gamma} \subset \bar{W}_u^\alpha \cap \bar{W}_s^\alpha$ in $X$. Therefore for all $\alpha$ near 0, $\Pi$ still has transversal intersections with both $\bar{W}_u^\alpha$ and $\bar{W}_s^\alpha$, which we denote by the smooth curves $(x_0^\alpha(\theta,\alpha),\theta(\text{mod } 2\pi))$ and $(x_0^s(\theta,\alpha),\theta(\text{mod } 2\pi))$ in $\Pi$, respectively. The split function $\beta(\theta,\alpha)$ is the signed “horizontal” (i.e. constant-$\theta$) distance between these curves in $\Pi$,

$$\beta(\theta,\alpha) = M(\theta,\alpha)/\|f_0^\perp(x^0(0))\|, \quad M(\theta,\alpha) = \langle f_0^\perp(x^0(0)), x_0^u(\theta,\alpha) - x_0^s(\theta,\alpha) \rangle.$$

Since the “Melnikov” function $M(\theta,\alpha)$ is a positive constant multiple of the split function, $\bar{W}_s^\alpha$ and $\bar{W}_u^\alpha$ intersect (along $\Pi$) if and only if

$$M(\theta,\alpha) = 0$$

for some $\theta \in S^1$.

For $\alpha = 0$ we have

$$M(\theta,0) = 0 \quad \text{for all } \theta \in S^1,$$

because of the coincidence of $\bar{W}_s^\alpha$ and $\bar{W}_u^\alpha$ along the homoclinic manifold $\bar{\Gamma}$, at its intersection with $\Pi$, and thus $M(\theta,\alpha) = O(|\alpha|)$. For $\alpha \neq 0$, we expand $M(\theta,\alpha)$ in a Taylor series in $\alpha$ at $\alpha = 0$, obtaining

$$M(\theta,\alpha) = \alpha M_\alpha(\theta,0) + O(|\alpha|^2).$$

Now we focus on finding a computable expression for the leading-order coefficient $M_\alpha(\theta,0)$. Fix $\alpha \neq 0$ sufficiently small, and fix $\theta_0 \in S^1$, and take two initial conditions for (4.5)

$$\tilde{x}(0) = (x_0^u(\theta_0,\alpha), \theta_0(\text{mod } 2\pi))$$

and

$$\tilde{x}(0) = (x_0^s(\theta_0,\alpha), \theta_0(\text{mod } 2\pi)),$$

in the intersections of $\bar{W}_u^\alpha$ and $\bar{W}_s^\alpha$ with $\Pi$. The two initial conditions generate, by solving the initial value problems for (4.5), two solutions

$$\tilde{x}(t) = \tilde{x}^u(t, \theta_0, \alpha) = (x^u(t,\theta_0,\alpha), \omega t + \theta_0(\text{mod } 2\pi))$$

and

$$\tilde{x}(t) = \tilde{x}^s(t, \theta_0, \alpha) = (x^s(t,\theta_0,\alpha), \omega t + \theta_0(\text{mod } 2\pi)),$$

where $x(t) = x^{u,s}(t,\theta_0,\alpha) \in \mathbb{R}^2$ are both solutions of the nonautonomous differential equation

$$\dot{x}(t) = f(t + \theta_0/\omega, x(t), \alpha) = f_0(x(t)) + \alpha f_1(x(t), \omega t + \theta_0).$$
For $\alpha = 0$ these two solutions are the same,

$$\dot{x}^u(t, \theta_0, 0) = x^0(t) = x^s(t, \theta_0, 0) \text{ in } \mathbb{R}^2.$$ 

Define two “time-dependent Melnikov” functions

$$M^u(t, \theta_0, \alpha) = \langle f_0^\perp(x^0(t)), x^u(t, \theta_0, \alpha) \rangle, \quad M^s(t, \theta_0, \alpha) = \langle f_0^\perp(x^0(t)), x^s(t, \theta_0, \alpha) \rangle,$$

so that when $t = 0$ we have

$$M^u(0, \theta_0, \alpha) - M^s(0, \theta_0, \alpha) = M(\theta_0, \alpha).$$

Now we expand each time-dependent Melnikov function in a Taylor series in $\alpha$, about $\alpha = 0$.

For example, the partial derivative of $M^u(t, \theta_0, \alpha)$ with respect to $\alpha$, at $\alpha = 0$ is

$$M^u_\alpha(t, \theta_0, 0) = \langle f_0^\perp(x^0(t)), x^u_\alpha(t, \theta_0, 0) \rangle.$$ 

Then taking the derivative with respect to $t$ we get

$$\dot{M}^u_\alpha(t, \theta_0, 0) = \langle f_0^\perp(x^0(t)) \dot{x}^0(t), x^u_\alpha(t, \theta_0, 0) \rangle + \langle f_0^\perp(x^0(t)), \dot{x}^u_\alpha(t, \theta_0, 0) \rangle.$$ 

Now $\dot{x}^0(t) = f_0(x^0(t))$, and (recalling Homework Assignment 2, question 1(b)) the derivative of the solution with respect to the parameter $\alpha$ satisfies a linear nonhomogeneous differential equation,

$$\dot{x}^u_\alpha(t, \theta_0, 0) = f_x(t + \theta_0/\omega, x^u(t, \theta_0, 0), 0) x^u_\alpha(t, \theta_0, 0) + f_\alpha(t + \theta_0/\omega, x^u(t, \theta_0, 0), 0)$$

$$= f_{0,x}(x^0(t)) x^u_\alpha(t, \theta_0, 0) + f_1(x^0(t), \omega t + \theta_0).$$

Substituting this last expression into the expression for $\dot{M}^u_\alpha(t, \theta_0, 0)$, we get

$$\dot{M}^u_\alpha(t, \theta_0, 0) = \langle f_{0,x}(x^0(t)) f(x^0(t)), x^u_\alpha(t, \theta_0, 0) \rangle$$

$$+ \langle f_0^\perp(x^0(t)), f_{0,x}(x^0(t)) x^u_\alpha(t, \theta_0, 0) + f_1(x^0(t), \omega t + \theta_0) \rangle.$$ 

This simplifies (Exercise) to

$$\dot{M}^u_\alpha(t, \theta_0, 0) = \text{div } f_0(x^0(t)) M^u_\alpha(t, \theta_0, 0) + \langle f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle.$$ 

For the Hamiltonian vector field $f_0$ we have $\text{div } f_0 = 0$, so

$$\dot{M}^u_\alpha(t, \theta_0, 0) = \langle f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle,$$

and by the Fundamental Theorem of Calculus, integrating from $t = -\infty$ to $t = 0$ gives

$$M^u_\alpha(0, \theta_0, 0) - \lim_{t \to -\infty} M^u_\alpha(t, \theta_0, 0) = \int_{-\infty}^0 \langle f_0^\perp(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle \, dt.$$ 

As $t \to -\infty$ we have $x^0(t) \to p_0^0$ with $f_0(p_0^0) = 0$, and $x^u_\alpha(t, \theta_0, 0) \to p_\alpha^0(t + \theta_0/\omega, 0)$ which is periodic and therefore bounded, thus

$$\lim_{t \to -\infty} M^u_\alpha(t, \theta_0, 0) = \lim_{t \to -\infty} \langle f_0^\perp(x_0(t)), x^u_\alpha(t, \theta_0, 0) \rangle = 0,$$
and we have
\[ M^u_\alpha(0, \theta_0, 0) = \int_{-\infty}^{0} \langle f^\perp_0(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle \, dt. \]

Similarly (Exercise),
\[ M^s_\alpha(0, \theta_0, 0) = -\int_{0}^{+\infty} \langle f^\perp_0(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle \, dt, \]
and finally
\[ M_\alpha(\theta_0, 0) = M^u_\alpha(0, \theta_0, 0) - M^s_\alpha(0, \theta_0, 0) = \int_{-\infty}^{+\infty} \langle f^\perp_0(x^0(t)), f_1(x^0(t), \omega t + \theta_0) \rangle \, dt. \]

Notice that since \( M(\theta, 0) = 0 \) for all \( \theta \) we can “factor out” \( \alpha \) from the Taylor series expansion
\[ M(\theta, \alpha) = \alpha M_\alpha(\theta, 0) + O(|\alpha|^2) = \alpha \tilde{M}(\theta, \alpha), \quad \tilde{M}(\theta, \alpha) = M_\alpha(\theta, 0) + O(|\alpha|), \]
thus \( \beta(\theta, \alpha) = 0, \alpha \neq 0 \) if and only if
\[ \tilde{M}(\theta, \alpha) = 0, \quad \alpha \neq 0. \]

If the two conditions (M.0) and (M.1) hold, the Implicit Function Theorem can be used to solve
\[ \tilde{M}(\theta, \alpha) = 0 \]
uniquely near \((\theta_0, 0)\) to give the solution \( \hat{\theta}(\alpha) = \theta_0 + O(|\alpha|) \), which which also corresponds to a zero of the split function \( \beta(\theta, \alpha) \) for \( \alpha \neq 0 \) sufficiently close to 0.

**Generalizations of Theorems 4.2 and 4.3.** Melnikov’s method can be generalized to higher dimensions, or where the unperturbed \((\alpha = \alpha_0)\) system is not necessarily Hamiltonian. For example, suppose we have a one-parameter family of \( n \)-dimensional autonomous ODEs
\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^1. \]

In the dimension \( n = 2 \) case considered for the Andronov-Leontovich theorem, if (SL.0.i), (SL.0.ii) and (SL.0.iii) hold and \( f \) is not Hamiltonian (if \( f \) is Hamiltonian then \( \text{div} f \equiv 0 \) and then (SL.2) cannot be satisfied), the appropriate Melnikov integral is
\[ M_\alpha(\alpha_0) = \int_{-\infty}^{+\infty} \langle \eta(t), f_\alpha(x^0(t), \alpha_0) \rangle \, dt, \]
where
\[ \eta(t) = e^{-\int_0^t \text{div} f(x^0(\tau), \alpha_0) \, d\tau} f^\perp_0(x^0(t), \alpha_0) \in \mathbb{R}^2, \]
and the condition on the derivative of the split function (SL.1) is equivalent to a more computable expression
\[ M_\alpha(\alpha_0) \neq 0. \quad (\text{SL.1}') \]
Also, in dimensions \( n \geq 2 \), under suitable conditions \( \eta(t) \) is the unique, up to a scalar multiple, bounded solution of the “adjoint variational equation”
\[ \dot{\eta} = -A(t)^T \eta, \quad \eta \in \mathbb{R}^n, \quad t \in \mathbb{R}, \]
where \( A(t) = f_x(x^0(t), \alpha_0) \) is the linearization of the vector field at the homoclinic solution, and \(^T\) denotes the matrix transpose (if \( n = 2 \) this characterization of \( \eta(t) \) reduces, up to a scalar multiple, to the expression for \( \eta(t) \) used above).
Transverse homoclinic points imply homoclinic tangles

If a map in $\mathbb{R}^n$ ($n \geq 2$) has one transverse homoclinic point (a point where the stable and unstable manifolds of a fixed point have a transversal intersection), this implies that the map also has much more complicated dynamics, such as a “homoclinic tangle”. The textbook by Wiggins (see the Course Outline) has many more details.

Chaos

A dynamical system restricted to an invariant set $\Lambda$, or the invariant set $\Lambda$ itself, is called chaotic if

(i) $\Lambda$ is compact,
(ii) the system is topologically transitive on $\Lambda$, and
(iii) the system has sensitive dependence on $\Lambda$.

Theorem 4.4. (Smale & Birkhoff) If a smooth map $x \mapsto f(x)$, $x \in \mathbb{R}^n$, $n \geq 2$, has a hyperbolic fixed point and a transversal intersection of its stable and unstable manifolds, then $f$ has infinitely many cycles, and an invariant set $\Lambda$ on which $f$ is chaotic.

Thus, Melnikov’s method (Theorem 4.2) can be used to show that a specific system has a chaotic invariant set (such as Example 4.A).