We now consider smooth \( m \)-parameter families of \( n \)-dimensional vector fields (or autonomous ODEs)
\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,\]
where \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is smooth. Here, the \( x \)-values are in \( n \)-dimensional state space \( \mathbb{R}^n \) and the \( \alpha \)-values are in \( m \)-dimensional parameter space \( \mathbb{R}^m \). For each fixed \( \alpha \), we have a vector field (an autonomous ODE), and the vector field changes smoothly as \( \alpha \) is changed. A parameter value \( \alpha_0 \) is a bifurcation value if for every open neighbourhood of \( \alpha_0 \) in \( \mathbb{R}^m \), there is always some \( \alpha_1 \) in that neighbourhood such that \( \dot{x} = f(x, \alpha_0) \) and \( \dot{x} = f(x, \alpha_1) \) are not topologically equivalent. A bifurcation diagram (or bifurcation set) is a parametric portrait (a stratification of parameter space induced by topological equivalence of flows in state space), together with the corresponding phase portraits in state space. A branching diagram is a diagram in parameter-state space \( \mathbb{R}^m \times \mathbb{R}^n \) showing branches of equilibria \( x = x^0_{[j]}(\alpha) \) and their stability (but not very practical if \( m > 1 \) or \( n > 1 \)). A local bifurcation is a bifurcation where the topological nonequivalence of flows occurs in some sufficiently small open neighbourhood of an equilibrium.

Similar definitions are made for smooth \( m \)-parameter families of \( n \)-dimensional maps
\[
x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.\]

Topological equivalence of families

We compare smooth families of vector fields and define more precisely what we mean when we say families have “qualitatively the same” dynamics for corresponding parameter values.

Two families
\[
\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m, \quad (3.1)
\]
and
\[
\frac{dy}{ds} = g(y, \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m, \quad (3.2)
\]
are topologically equivalent if there is a homeomorphism of parameter variables \( p : \mathbb{R}^m \to \mathbb{R}^m, \beta = p(\alpha) \), and a family of homeomorphisms of state variables \( h(\cdot, \alpha) : \mathbb{R}^n \to \mathbb{R}^n, y = h(x, \alpha) \) that maps the orbits of (3.1) at parameter values \( \alpha \) onto the orbits of (3.2) at parameter values \( \beta = p(\alpha) \), preserving the orientation of time. The two families are locally topologically equivalent if \( p \) and \( h(\cdot, \alpha) \) are local homeomorphisms, defined on open subsets of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively (typically, on some sufficiently small open neighbourhoods of specific points).

Similarly, we can define topological equivalence and local topological equivalence for two families of maps,
\[
x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,\]
and
\[
y \mapsto g(y, \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m.\]

From now on, we consider one-parameter families, \( m = 1 \).
Fold bifurcation for one-parameter families of one-dimensional vector fields

Let \( f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) be locally defined and sufficiently smooth near \((x_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1\), and consider the one-parameter family of one-dimensional vector fields

\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1.
\]

For a local bifurcation, we look for an equilibrium \( x_0^0 \) that is nonhyperbolic for some specific parameter value \( \alpha_0 \). In a one-dimensional vector field, there is only one way for an equilibrium to be nonhyperbolic: \( \lambda_0 = 0 \), where \( \lambda_0 = f_x(x_0^0, \alpha_0) \) is the eigenvalue of the \( 1 \times 1 \) linearization \( \dot{\xi} = f_x(x_0^0, \alpha_0)\xi \). The two conditions

\[
f(x_0^0, \alpha_0) = 0 \quad \text{(equilibrium),} \tag{F.0.i}
\]

\[
f_x(x_0^0, \alpha_0) = 0 \quad \text{(bifurcation),} \tag{F.0.ii}
\]

give, for \( \alpha = \alpha_0 \), an equilibrium for (\( \dagger \)) that exists, by (F.0.i), and is nonhyperbolic, by (F.0.ii). At this point, we have a potential bifurcation, since the phase portrait near a nonhyperbolic equilibrium is, typically, topologically sensitive to arbitrarily small perturbations of the vector field \( f(x, \alpha_0) \).

Expanding \( f(x, \alpha) \) at any \( \alpha \) in a Taylor series about \( x_0^0 \), we have

\[
\dot{x} = f(x_0^0, \alpha) + f_x(x_0^0, \alpha)(x - x_0^0) + \frac{1}{2}f_{xx}(x_0^0, \alpha)(x - x_0^0)^2 + O(|x - x_0^0|^3).
\]

Then expanding each \( \alpha \)-dependent coefficient about \( \alpha_0 \) and using (F.0.i) and (F.0.ii), we have \( f_x(x_0^0, \alpha) = O(|\alpha - \alpha_0|) \) and

\[
f(x_0^0, \alpha) = f(x_0^0, \alpha_0)(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2), \quad \frac{1}{2}f_{xx}(x_0^0, \alpha) = \frac{1}{2}f_{xx}(x_0^0, \alpha_0) + O(|\alpha - \alpha_0|).
\]

For a fold bifurcation (also known as a saddle-node bifurcation), we also assume that two “generic” conditions hold:

\[
a = f_{\alpha}(x_0^0, \alpha_0) \neq 0 \quad \text{(transversality),} \tag{F.1}
\]

\[
b = \frac{1}{2}f_{\alpha\alpha}(x_0^0, \alpha_0) \neq 0 \quad \text{(nondegeneracy).} \tag{F.2}
\]

Given that the family (\( \dagger \)) satisfies the equalities (F.0.i) and (F.0.ii), the inequalities (F.1) and (F.2) are satisfied “generically” – roughly speaking, if we choose an “arbitrary” family (\( \dagger \)) that satisfies (F.0.i) and (F.0.ii), then it is “highly probable” (in some sense that can be made mathematically precise, but not worth taking the time to cover the required background) that it also satisfies (F.1) and (F.2). If some model doesn’t satisfy the generic conditions at a potential bifurcation, then the potential bifurcation is not a fold bifurcation. It may be some other kind of bifurcation, or even in some pathological cases, not a bifurcation at all. If this happens, there is often some previously overlooked “exceptional” feature of the model, like a constraint, or symmetry, or other special structure (e.g. Hamiltonian, or orbitally equivalent to Hamiltonian), that explains why the generic conditions are not satisfied.

Expanding \( f(x, \alpha) \) in a Taylor series about \((x_0^0, \alpha_0)\) and using (F.0.i)–(F.2), the family (\( \dagger \)) is

\[
\dot{x} = a(\alpha - \alpha_0) + b(x - x_0^0)^2 + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - x_0^0| + |x - x_0^0|^3).
\]

It turns out that if \( a \neq 0 \) and \( b \neq 0 \), then the higher-order terms

\[
O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - x_0^0| + |x - x_0^0|^3)
\]

do not qualitatively affect the local dynamics, and so can be ignored. More precisely, it can be proved that if \( a \neq 0 \) and \( b \neq 0 \), then (\( \dagger \)) is locally topologically equivalent to the normal form family which is obtained by discarding (or truncating) the specified higher-order terms in the Taylor series. (NOTE: Which specific terms are considered “higher-order” and therefore can be ignored depends on the bifurcation being studied!)
Theorem 3.1. If \( f : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) is \( C^3 \) in an open set containing \((x_0^0, \alpha^0)\) and satisfies the four conditions (F.0.i)–(F.2), then the family
\[
\frac{dx}{dt} = f(x, \alpha) \quad \text{at} \quad (x_0^0, \alpha_0)
\]
has a fold bifurcation, locally topologically equivalent to the normal form
\[
\frac{dy}{ds} = a\beta + by^2 \quad \text{at} \quad (0, 0).
\]

By rescaling the variables \( \beta = p(\alpha) = \alpha - \alpha_0 + O(|\alpha - \alpha_0|^2) \) and \( y = h(x, \alpha) = x - x_0^0 + O(|\alpha - \alpha_0| + |x - x_0^0|^2) \), and possibly changing the sign of \( \beta \), we could choose the new scalings so that \( a = 1 \) and \( b = \pm 1 \), and then the normal form family can be expressed even more simply as a representative of one of only two distinct possibilities under local topological equivalence of families,
\[
\frac{dy}{ds} = \beta \pm y^2.
\]
The textbook calls this the “topological” normal form.

After the higher-order terms have been “transformed away” with a suitable change of variables, the dynamics of the normal form are easily determined.

If the family satisfies a constraint condition, so that it must have an equilibrium for all parameter values, then it might have a transcritical bifurcation (see Homework Assignment 3).

Symmetric pitchfork bifurcation for one-parameter families of one-dimensional vector fields

Many models have built-in symmetries, and these can change the types of bifurcations that are generic. A simple case of symmetry occurs when each member of a family of one-dimensional vector fields is an odd function of the state variable. Consider the family
\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1.
\]
If the family satisfies the symmetry condition that \( f \) is odd in \( x \) for all \( \alpha \),
\[
f(-x, \alpha) = -f(x, \alpha) \quad \text{for all} \quad x, \alpha \quad \text{(symmetry) (SP.0.i)}
\]
(one of two possible types of “\( \mathbb{Z}_2 \)-equivariance”), then as a consequence of (SP.0.i) we have
\[
f(0, \alpha) = 0 \quad \text{for all} \quad \alpha \quad \text{(*)}
\]
(i.e. \( x_{[1]}^0(\alpha) \equiv 0 \) is an equilibrium). Then (*) implies that we can “factor out” \( x \) from \( f(x, \alpha) \) and write
\[
f(x, \alpha) = x\tilde{f}(x, \alpha),
\]
where \( \tilde{f} \) is smooth at \((0, \alpha)\). Furthermore, the symmetry condition (SP.0.i) implies that \( \tilde{f} \) is even in \( x \), and that all even-order partial derivatives of \( \tilde{f} \) with respect to \( x \) must vanish at \( x = 0 \), for any \( \alpha \):
\[
f_{xx}(0, \alpha) = 0, \quad f_{xxxx}(0, \alpha) = 0, \quad \ldots.
\]
If, in addition, at some specific parameter value \( \alpha_0 \), the equilibrium \( x = 0 \) is nonhyperbolic:
\[
f_x(0, \alpha_0) = 0 \quad \text{(bifurcation), (SP.0.ii)}
\]
then we have
\[
\dot{x} = f(x, \alpha) = x\tilde{f}(x, \alpha) = x \left[ f_x(0, \alpha) + \frac{1}{6}f_{xxx}(0, \alpha)x^2 + O(|x|^4) \right] \quad \text{(odd in} \ x),
\]
where, expanding in $\alpha - \alpha_0$ and using (SP.0.ii),

$$f_x(0, \alpha) = \frac{f_{x0}(0, \alpha_0)}{a}(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2), \quad \frac{1}{6}f_{xxx}(0, \alpha) = \frac{1}{6}f_{xxx}(0, \alpha_0) + O(|\alpha - \alpha_0|),$$

We assume two (generic) conditions hold, that the leading-order coefficients in the above expansions do not vanish:

$$a = f_{x0}(0, \alpha_0) \neq 0 \quad (\text{transversality}), \quad (\text{SP.1})$$

$$b = \frac{1}{6}f_{xxx}(0, \alpha_0) \neq 0 \quad (\text{nondegeneracy}). \quad (\text{SP.2})$$

Then the family has a symmetric pitchfork bifurcation at $(0, \alpha_0)$. One branch of equilibria $x^0_{[i]}(\alpha) \equiv 0$ is already known from the symmetry, and solving $\dot{f}(x, \alpha) = 0$, we obtain two more branches of nonzero equilibria $x^0_{[j]}(\alpha)$, $j = 2, 3$, for $\alpha$ on one side of $\alpha_0$ (which side, $\alpha < \alpha_0$ or $\alpha > \alpha_0$, is called the “direction” of the bifurcation), $\alpha$ sufficiently near $\alpha_0$.

**Theorem 3.2.** If $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is $C^5$ in an open set containing $(0, \alpha_0)$ and satisfies the four conditions (SP.0.i)–(SP.2), then the family

$$\frac{dx}{dt} = f(x, \alpha) \quad \text{at } (0, \alpha_0)$$

has a symmetric pitchfork bifurcation, locally topologically equivalent to the normal form

$$\frac{dy}{ds} = a\beta y + by^3 \quad \text{at } (0, 0).$$

It is easy to verify that the stability of the bifurcating nonzero equilibria depends on the sign of $b$, and the “direction” of bifurcation depends on the sign of $ab$. If $b < 0$ (and therefore the bifurcating nonzero equilibria are stable), the pitchfork bifurcation is called “supercritical”; if $b > 0$ the pitchfork bifurcation is called “subcritical”.

Fold bifurcation for one-parameter families of one-dimensional maps

If a one-parameter family of one-dimensional maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,$$

has, for some critical parameter value $\alpha_0$, a nonhyperbolic fixed point $x^0_0$, then in one state space dimension, this could occur only if $|\mu_0| = 1$, where $\mu_0 = f_x(x^0_0, \alpha_0)$ is the multiplier of the $1 \times 1$ linearization for $\alpha = \alpha_0$, therefore $\mu_0 = \pm 1$ since $\mu_0$ is real. We first consider the case $\mu_0 = +1$. If $f(x, \alpha)$ satisfies

$$f(x^0_0, \alpha_0) = x^0_0 \quad (\text{fixed point}), \quad (\text{FM.0.i})$$

$$f_x(x^0_0, \alpha_0) = 1 \quad (\text{bifurcation}), \quad (\text{FM.0.ii})$$

$$a = f_\alpha(x^0_0, \alpha_0) \neq 0 \quad (\text{transversality}), \quad (\text{FM.1})$$

$$b = \frac{1}{2}f_{xx}(x^0_0, \alpha_0) \neq 0 \quad (\text{nondegeneracy}), \quad (\text{FM.2})$$

then the family has a fold bifurcation for maps. The Taylor series of the family $x \mapsto f(x, \alpha)$ at $(x^0_0, \alpha_0)$ is

$$x \mapsto x + a(\alpha - \alpha_0) + b(x - x^0_0)^2 + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - x^0_0| + |x - x^0_0|^3),$$

and it can be proved that the higher-order terms do not qualitatively affect the local dynamics. There is a change of variables (local homeomorphisms of parameter and state variables) that “transforms away” the higher-order terms:
Theorem 3.3. If \( f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) is \( C^3 \) in an open set containing \((x_0^0, \alpha_0)\) and satisfies the four conditions (FM.0.i)–(FM.2), then the family
\[
x \mapsto f(x, \alpha) \quad \text{at} \quad (x_0^0, \alpha_0)
\]
has a fold bifurcation, locally topologically equivalent to the normal form
\[
y \mapsto y + a\beta + by^2 \quad \text{at} \quad (0,0).
\]

Families of maps can also have transcritical bifurcations and symmetric pitchfork bifurcations associated with a critical multiplier \( \mu_0 = +1 \) of the linearization. Like the fold bifurcations for maps, the transcritical and symmetric pitchfork bifurcations for maps behave essentially like discrete-time analogues of the corresponding bifurcations for flows.

Flip (or period doubling) bifurcation for one-parameter families of one-dimensional maps

Now we consider the more interesting case \( \mu_0 = -1 \). Consider a one-parameter family of one-dimensional maps
\[
x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,
\]
where \( f \) is smooth near a point \((x_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1\). If the family satisfies
\[
\begin{align*}
f(x_0^0, \alpha_0) &= x_0^0 \quad \text{(fixed point),} \\
f_x(x_0^0, \alpha_0) &= -1 \quad \text{(bifurcation),}
\end{align*}
\]
then \( x_0^0 \) is a fixed point that exists for \( \alpha = \alpha_0 \), and it is nonhyperbolic. In this case, the Implicit Function Theorem can be used to solve \( f(x, \alpha) - x = 0 \) to obtain a locally defined, locally unique, smooth solution \( x = x^0(\alpha) \) with \( x^0(\alpha_0) = x_0^0 \), a smooth curve \((x^0(\alpha), \alpha)\) of fixed points (i.e. \( f(x^0(\alpha), \alpha) = x^0(\alpha) \)) through \((x_0^0, \alpha_0)\), so the number of local fixed points does not change if we vary \( \alpha \) near \( \alpha_0 \).

We change coordinates with a shift
\[
x = x^0(\alpha) + u
\]
so that for any \( \alpha, u = 0 \) corresponds to the fixed point \( x = x^0(\alpha) \) and the family is transformed into
\[
u \mapsto \mu(\alpha) u + \hat{f}(u, \alpha) = \mu(\alpha) u + \hat{f}_2(\alpha) u^2 + \hat{f}_3(\alpha) u^3 + O(|u|^4),
\]
with fixed point \( u = 0 \), where
\[
\mu(\alpha) = f_x(x^0(\alpha), \alpha) = \mu(\alpha_0) + \mu'(\alpha_0)(\alpha - \alpha_0) + O(|\alpha - \alpha_0|^2), \quad \mu(\alpha_0) = -1,
\]
\[
\hat{f}_2(\alpha) = \frac{1}{2} f_{xx}(x^0(\alpha), \alpha) = \frac{1}{2} f_{xx}(x_0^0, \alpha_0) + O(|\alpha - \alpha_0|),
\]
\[
\hat{f}_3(\alpha) = \frac{1}{6} f_{xxx}(x^0(\alpha), \alpha) = \frac{1}{6} f_{xxx}(x_0^0, \alpha_0) + O(|\alpha - \alpha_0|).
\]

We assume a (generic) condition
\[
a = -\mu'(\alpha_0) = - \frac{d}{d\alpha} [f_x(x^0(\alpha), \alpha)]_{\alpha=\alpha_0} \neq 0,
\]
which implies that the linearized stability of the fixed point \( x^0(\alpha) \) changes (because the value of the multiplier \( \mu(\alpha) \) passes through \( \mu(\alpha_0) = -1 \) with nonzero “speed”), as \( \alpha \) increases through \( \alpha_0 \). Using an equivalent (Exercise: show it is equivalent) but more practically calculated expression for \( \mu'(\alpha_0) \), we assume
\[
a = -f_{x\alpha}(x_0^0, \alpha_0) - \frac{1}{2} f_{xx}(x_0^0, \alpha_0) f_{\alpha}(x_0^0, \alpha_0) \neq 0 \quad \text{(transversality).}
\]
Next, a coordinate change applied to \( u \mapsto \mu(\alpha)u + \hat{f}(u, \alpha) \), of the form
\[
u = h(w, \alpha) = w + \delta(\alpha)w^2;
\]
can be found so that in the transformed family the quadratic Taylor series coefficient in the state variable (i.e. the coefficient of \( w^2 \)) is “removed” for all \( \alpha \) (see Homework Assignment 4), and the transformed family is then “simplified”:
\[
w \mapsto \mu(\alpha)w + g_3(\alpha)w^3 + O(|w|^4).
\]
The idea is to choose the quadratic coefficient in the coordinate change \( \delta(\alpha) \) carefully, so that in the transformed family we have the quadratic coefficient \( g_2(\alpha) = 0 \) for all \( \alpha \). If this specific choice of \( \delta(\alpha) \) is made, then it turns out that the cubic coefficient (i.e. the coefficient of \( w^3 \)) in the transformed family becomes
\[
g_3(\alpha) = \hat{f}_3(\alpha_0) + \left[ \hat{f}_2(\alpha_0) \right]^2 + O(|\alpha - \alpha_0|).\]
We now assume another generic condition, that for \( \alpha = \alpha_0 \) this cubic coefficient \( g_3(\alpha_0) \) does not vanish:
\[
b = \frac{1}{6} f_{xxx}(x_0^0, \alpha_0) + \left[ \frac{1}{2} f_{xx}(x_0^0, \alpha_0) \right]^2 \neq 0 \quad (\text{nondegeneracy}). \tag{PD.2}
\]
Then it can be proved that the family of maps has a flip bifurcation (also called a period doubling bifurcation) at \( (x_0^0, \alpha_0) \):

**Theorem 3.4.** If \( g : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) is \( C^4 \) in an open set containing \( (x_0^0, \alpha_0) \) and satisfies the four conditions (PD.0.1)–(PD.2), then the family
\[
x \mapsto f(x, \alpha) \quad \text{at} \ (x_0^0, \alpha_0)
\]
has a flip (or period doubling) bifurcation, locally topologically equivalent to the normal form
\[
y \mapsto -y - a\beta y + by^3 \quad \text{at} \ (0, 0).
\]
The second iterate of the map, \( x \mapsto f^2(x, \alpha) = f(f(x, \alpha), \alpha) \) is locally topologically equivalent to the second iterate of the normal form
\[
y \mapsto y + 2a\beta y - 2by^3 + O(|\beta|^2|y| + |\beta||y|^3 + |y|^5),
\]
which has bifurcating nonzero fixed points. These nonzero fixed points for the second iterate correspond to points on the orbits of bifurcating nontrivial 2-cycles for the normal form, and also for the original map \( x \mapsto f(x, \alpha) \). Their stability depends on the sign of \( b \), the direction of bifurcation depends on the sign of \( ab \).

**Poincaré normal forms**

For vector fields or maps, there is a systematic procedure to “simplify” the lower-order terms of the Taylor series at an equilibrium or a fixed point. Furthermore, the procedure can be modified for families. We introduced this procedure above, for families of one-dimensional maps. Here we develop this procedure for \( n \)-dimensional vector fields.

Let \( H_k \) denote the finite-dimensional vector space of all vector fields : \( \mathbb{R}^n \to \mathbb{R}^n \) whose components are all homogeneous polynomials of order \( k \). Assume that \( 0 \in \mathbb{R}^n \) is an equilibrium of \( \dot{x} = f(x) \), \( x \in \mathbb{R}^n \), and expand in a Taylor series at the equilibrium 0,
\[
\dot{x} = Ax + \hat{f}(x) = Ax + \hat{f}^{(2)}(x) + \hat{f}^{(3)}(x) + \cdots, \tag{3.3}
\]
where \( Ax = f_x(0)x \in H_1 \) is the linearization of the vector field at the equilibrium, and \( \hat{f}^{(k)}(x) \in H_k \) are the terms in the Taylor series of order exactly \( k \), for \( k = 2, 3, \ldots \).
Let us assume that the order 1 terms \( Ax \) have already been “simplified” by choosing coordinates \( x \) so that \( A \) is in real normal form, and we can start by “simplifying” the order \( k = 2 \) terms. Introduce a coordinate change (a local diffeomorphism) written as

\[
x = y + h^{(2)}(y)
\]

where \( h^{(2)} \in H_2 \). Applying any such coordinate change, the linearization is unchanged and the vector field (3.3) is transformed into the (locally topologically equivalent) vector field

\[
\dot{y} = Ay - (L_A h^{(2)})(y) + \hat{f}^{(2)}(y) + O(\|y\|^3),
\]

where for any integer \( k \geq 2 \) the linear operator \( L_A : H_k \to H_k \) is defined by

\[
(L_A h^{(k)})(y) = h_y^{(k)}(y)Ay - Ah^{(k)}(y).
\]

For \( k = 2 \), we find the range \( L_A(H_2) \) and a complementary subspace \( \tilde{H}_2 \) (not necessarily unique) in \( H_2 \) so that

\[
H_2 = L_A(H_2) \oplus \tilde{H}_2.
\]

Relative to any specific, fixed complementary subspace \( \tilde{H}_2 \), every \( \hat{f}^{(2)} \in H_2 \) has a unique decomposition

\[
\hat{f}^{(2)}(y) = g^{(2)}(y) + r^{(2)}(y), \quad g^{(2)} \in L_A(H_2), \quad r^{(2)} \in \tilde{H}_2.
\]

Then since \( g^{(2)} \) belongs to the range of \( L_A \) in \( H_2 \), there exists some specific \( h^{(2)} \in H_2 \) such that the coordinate change “removes” the \( L_A(H_2) \)-component of \( \hat{f}^{(2)} \),

\[
-(L_A h^{(2)})(y) + g^{(2)}(y) = 0,
\]

and therefore transforms (3.3) into

\[
\dot{y} = Ay + r^{(2)}(y) + O(\|y\|^3). \tag{3.4}
\]

In this way the vector field (3.3) is “simplified” by the coordinate change that “removes” as many coefficients of order 2 as possible. The term \( r^{(2)}(y) \) (if nonzero) contains only resonant terms of order 2, and we say the vector field has been put into \textbf{Poincaré normal form up to order 2}. We also say that (3.4) is the Poincaré normal form up to order 2, of (3.3). Using induction, one can show that if \( f \) is smooth enough, one may transform the vector field (3.3) into Poincaré normal form up to any finite order \( m, m \geq 2 \).

**Theorem 3.5.** If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^{m+1} \) \( m \geq 2 \) in an open set containing 0 and \( f(0) = 0 \), then there exists a coordinate change

\[
x = y + h^{(2)}(y) + \cdots + h^{(m)}(y), \quad h^{(k)} \in H_k, \quad k = 2, \cdots, m,
\]

that transforms

\[
\dot{x} = f(x) = Ax + \hat{f}^{(2)}(x) + \cdots + \hat{f}^{(m)}(x) + O(\|x\|^{m+1}), \quad \hat{f}^k \in H_k, \quad k = 2, \cdots, m,
\]

into the (locally topologically equivalent) Poincaré normal form up to order \( m \),

\[
\dot{y} = Ay + r^{(2)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \quad r^{(k)} \in \tilde{H}_k, \quad k = 2, \cdots, m,
\]

where each \( r^{(k)} \) contains only resonant terms of order \( k \), \( k = 2, \cdots, m \).

A similar procedure can be done for \( n \)-dimensional maps, and for families of \( n \)-dimensional vector fields or maps.

(Example 3.B – the Poincaré normal form of a nonhyperbolic “linear centre” – was done in class.)
A Hopf bifurcation is the bifurcation of limit cycles in a family of vector fields, associated with a pair of purely imaginary eigenvalues for the linearization at an equilibrium.

We review (more background was given in class) how to determine the existence and stability of limit cycles, at a Hopf bifurcation in a one-parameter family

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}. \]  

(3.5)

If there exist \( x_0^0 \in \mathbb{R}^2 \) and \( \alpha_0 \in \mathbb{R}^1 \) such that the conditions

\[ f(x_0^0, \alpha_0) = 0 \quad (\text{equilibrium}), \]

(H.0.i)

\[ A_0 = f_x(x_0^0, \alpha_0) \text{ has eigenvalues } \pm i \omega_0, \quad \omega_0 > 0 \quad (\text{bifurcation}), \]

(H.0.ii)

are satisfied, then \( x = x_0^0 \) is an equilibrium that is nonhyperbolic for \( \alpha = \alpha_0 \). The Implicit Function Theorem can be used to obtain a locally defined, locally unique, smooth curve \((x^0(\alpha), \alpha)\) of equilibria through \((x_0^0, \alpha_0)\). Make the first coordinate change

\[ x = x^0(\alpha) + u \]  

(I)

to transform the family (3.5) into

\[ \dot{u} = A(\alpha)u + \hat{f}(u, \alpha), \quad u \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}, \]  

(3.6)

where

\[ A(\alpha) = f_x(x^0(\alpha), \alpha), \quad \hat{f}(u, \alpha) = f(x^0(\alpha) + u, \alpha) - A(\alpha)u = O(\|u\|^2), \]

so that \( u = 0 \) is now the equilibrium for all \( \alpha \) near \( \alpha_0 \). For each \( \alpha \) near \( \alpha_0 \), the matrix \( A(\alpha) \) of the linearization has complex conjugate eigenvalues depending smoothly on \( \alpha \),

\[ \lambda_1(\alpha) = \mu(\alpha) + i \omega(\alpha), \quad \lambda_2(\alpha) = \bar{\lambda}_1(\alpha) = \mu(\alpha) - i \omega(\alpha), \]

where

\[ \mu(\alpha) = \text{Re} \lambda_1(\alpha) = \frac{1}{2} \text{tr} A(\alpha), \quad \omega(\alpha) = \text{Im} \lambda_1(\alpha) > 0 \]

with

\[ \mu(\alpha_0) = 0, \quad \omega(\alpha_0) = \omega_0 > 0. \]

We assume that when \( \alpha \) increases past \( \alpha_0 \), the eigenvalues cross the imaginary axis with nonzero “speed”,

\[ a = \mu'(\alpha_0) \neq 0 \quad (\text{transversality}), \]  

(H.1)

so that the linearized stability of the equilibrium \( x^0(\alpha) \) changes, as \( \alpha \) increases through \( \alpha_0 \).

We now set \( \alpha = \alpha_0 \) in (3.6) to check the nondegeneracy condition (H.2), below. For \( \alpha = \alpha_0 \) the vector field (3.6) is the same as in Example 3.B, done in class, and it was shown that it can be transformed into Poincaré normal form up to order 3. We summarize the projection method procedure to find the “cubic normal form coefficient”. Let \( A_0 = A(\alpha_0) \) and find a (complex) eigenvector \( q \),

\[ A_0 q = i \omega_0 q, \quad q = \left( \begin{array}{c} q_1 \\ q_2 \end{array} \right) \in \mathbb{C}^2, \]

and also find an adjoint eigenvector \( p \),

\[ A_0^\top p = -i \omega_0 p, \quad p = \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) \in \mathbb{C}^2, \]
normalized so that
\[ \langle p, q \rangle = 1, \]
where the angle brackets denote the inner product in \( \mathbb{C}^2 \), \( \langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2 \). Recall we automatically have
\[ \langle p, \bar{q} \rangle = 0. \]

Then any \( u \in \mathbb{R}^2 \) can be written uniquely as
\[ u = z_1 q + \bar{z}_1 \bar{q}, \quad z_1 = \langle p, u \rangle \in \mathbb{C} \]
(i.e. \( u_1 = z_1 q_1 + \bar{z}_1 \bar{q}_1 \), \( u_2 = z_1 q_2 + \bar{z}_1 \bar{q}_2 \) for the components). Substitute this into (3.6) and take the inner product with the adjoint eigenvector \( p \) to obtain the equation for \( \dot{z}_1 \),
\[ \dot{z}_1 = i \omega_0 z_1 + \frac{1}{2} \bar{q}(z_1 q + \bar{z}_1 \bar{q}, \alpha_0) \cdot g(z_1, \bar{z}_1, \alpha_0) = O(|z_1|^2) \quad (3.7) \]
Expand the nonlinear part in powers of \( z_1 \) and \( \bar{z}_1 \) (treating \( z_1, \bar{z}_1 \) as independent variables),
\[ \langle p, \bar{q}(z_1 q + \bar{z}_1 \bar{q}, \alpha_0) \rangle = \int_{\mathbb{R}} \bar{g}(\alpha_0) \bar{g}_2(\alpha_0) \bar{g}_1(\alpha_0) z_1 \bar{z}_1 + \frac{1}{2} g(\alpha_0) z_1^2 + \frac{1}{3} g(\alpha_0) \bar{z}_1^2 + \frac{1}{2} \bar{g}(\alpha_0) \bar{g}_3(\alpha_0) \bar{g}_2(\alpha_0) \bar{g}_1(\alpha_0) \bar{z}_1^2 \bar{z}_1 + \cdots. \]
Identify the coefficients of \( z_1^2, \bar{z}_1^2 \) and \( z_1 \bar{z}_1 \) in this expansion, and assume a nonvanishing cubic normal form coefficient
\[ b = \text{Re} \left[ \int_{\mathbb{R}} \bar{g}(\alpha_0) \bar{g}_2(\alpha_0) \bar{g}_1(\alpha_0) \right] \neq 0 \quad \text{(nondegeneracy).} \quad (H.2) \]
Then the original family (3.5) has a \textbf{Hopf} bifurcation at \((x_0^0, \alpha_0)\):

**Theorem 3.6.** If \( f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \) is \( C^4 \) in an open set containing \((x_0^0, \alpha_0)\) and satisfies the four conditions (H.0.i)–(H.2), then the family
\[ \frac{dx}{dt} = f(x, \alpha) \quad \text{at} \quad (x_0^0, \alpha_0) \]
has a Hopf bifurcation, locally topologically equivalent to the normal form
\[ \begin{align*}
\frac{dy_1}{ds} &= a \beta y_1 - \omega_0 y_2 + b(y_1^2 + y_2^2)y_1 \\
\frac{dy_2}{ds} &= \omega_0 y_1 + a \beta y_2 + b(y_1^2 + y_2^2)y_2
\end{align*} \quad \text{at} \quad ((0,0),0). \]
In polar coordinates \( y_1 = \rho \cos(\phi), y_2 = \rho \sin(\phi) \), the normal form is
\[ \begin{align*}
\dot{\rho} &= a \beta \rho + b \rho^3, \\
\dot{\phi} &= \omega_0, \\
(\rho, \phi) &\in (0, \infty) \times S^1,
\end{align*} \]
which is easily studied to determine the existence and stability of limit cycles. One can verify (Exercise: in class we considered \( a > 0 \) and \( b < 0 \); for the three remaining possibilities of \( a \neq 0 \) and \( b \neq 0 \) determine the bifurcation diagrams and branching diagrams) there exist limit cycles bifurcating from the equilibrium at \( \beta = 0 \), which are stable or unstable depending on the sign of \( b \). The direction of bifurcation (i.e. whether the limit cycles exist only for \( \beta > 0 \) or only for \( \beta < 0 \) depends on the sign of \( ab \).