3. One-Parameter Local Bifurcations – Notes
2018 October 15 – as of October 15, 2018

Bifurcations

(Example 3.A done in class.)

We now consider smooth \( m \)-parameter families of \( n \)-dimensional vector fields (or autonomous ODEs)

\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,
\]

where \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is smooth. Here, the \( x \)-values are in \( n \)-dimensional state space \( \mathbb{R}^n \) and the \( \alpha \)-values are in \( m \)-dimensional parameter space \( \mathbb{R}^m \). For each fixed \( \alpha \), we have a vector field (an autonomous ODE), and the vector field changes smoothly as \( \alpha \) is changed. A parameter value \( \alpha_0 \) is a bifurcation value if for every open neighbourhood of \( \alpha_0 \) in \( \mathbb{R}^m \), there is always some \( \alpha_1 \) in that neighbourhood such that \( \dot{x} = f(x, \alpha_0) \) and \( \dot{x} = f(x, \alpha_1) \) are not topologically equivalent. A bifurcation diagram (or bifurcation set) is a parametric portrait (a stratification of parameter space induced by topological equivalence of flows in state space), together with the corresponding phase portraits in state space. A branching diagram is a diagram in parameter-state space \( \mathbb{R}^m \times \mathbb{R}^n \) showing branches of equilibria \( x = x^0_{[\alpha]}(\alpha) \) and their stability (but not very practical if \( m > 1 \) or \( n > 1 \)). A local bifurcation is a bifurcation where the topological nonequivalence of flows occurs in some sufficiently small open neighbourhood of an equilibrium.

Similar definitions are made for smooth \( m \)-parameter families of \( n \)-dimensional maps

\[
 x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.
\]

Topological equivalence of families

We compare smooth families of vector fields and define more precisely what we mean when we say families have “qualitatively the same” dynamics for corresponding parameter values.

Two families

\[
\frac{dx}{dt} = f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,
\]

and

\[
\frac{dy}{ds} = g(y, \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m,
\]

are topologically equivalent if there is a homeomorphism of parameter variables \( p : \mathbb{R}^m \to \mathbb{R}^m, \beta = p(\alpha) \), and a family of homeomorphisms of state variables \( h(\cdot, \alpha) : \mathbb{R}^n \to \mathbb{R}^n, y = h(x, \alpha) \) that maps the orbits of (3.1) at parameter values \( \alpha \) onto the orbits of (3.2) at parameter values \( \beta = p(\alpha) \), preserving the orientation of time. The two families are locally topologically equivalent if \( p \) and \( h(\cdot, \alpha) \) are local homeomorphisms, defined on open subsets of \( \mathbb{R}^m \) and \( \mathbb{R}^n \), respectively (typically, on some sufficiently small open neighbourhoods of specific points).

Similarly, we can define topological equivalence and local topological equivalence for two families of maps,

\[
x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m,
\]

and

\[
y \mapsto g(y, \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m.
\]

From now on, we consider one-parameter families, \( m = 1 \).
Fold bifurcation for one-parameter families of one-dimensional vector fields

Let \( f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) be locally defined and sufficiently smooth near \((x_0^0, \alpha_0) \in \mathbb{R}^1 \times \mathbb{R}^1\), and consider the one-parameter family of one-dimensional vector fields

\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1.
\]

(*)

For a local bifurcation, we look for an equilibrium \(x_0^0\) that is nonhyperbolic for some specific parameter value \(\alpha_0\). In a one-dimensional vector field, there is only one way for an equilibrium to be nonhyperbolic: \(\lambda_0 = 0\), where \(\lambda_0 = f_x(x_0^0, \alpha_0)\) is the eigenvalue of the \(1 \times 1\) linearization \(\dot{\xi} = f_x(x_0^0, \alpha_0)\xi\). The two conditions

\[
f(x_0^0, \alpha_0) = 0, \quad (equilibrium) \quad \text{(F.0.i)}
\]

\[
f_x(x_0^0, \alpha_0) = 0, \quad (bifurcation) \quad \text{(F.0.ii)}
\]

give, for \(\alpha = \alpha_0\), an equilibrium for (*) that exists, by (F.0.i), and is nonhyperbolic, by (F.0.ii). At this point, we have a potential bifurcation, since the phase portrait near a nonhyperbolic equilibrium is, typically, topologically sensitive to arbitrarily small perturbations of the vector field \(f(x, \alpha_0)\). For a fold bifurcation (also known as a saddle-node bifurcation), we also assume that two “generic” conditions hold:

\[
a = f_\alpha(x_0^0, \alpha_0) \neq 0, \quad (transversality) \quad \text{(F.1)}
\]

\[
b = \frac{1}{2} f_{xx}(x_0^0, \alpha_0) \neq 0. \quad (nondegeneracy) \quad \text{(F.2)}
\]

Given that the family (*) satisfies the equalities (F.0.i) and (F.0.ii), the inequalities (F.1) and (F.2) are satisfied “generically” – roughly speaking, if we choose an “arbitrary” family (*) that satisfies (F.0.i) and (F.0.ii), then it is “highly probable” (in some sense that can be made mathematically precise, but not worth taking the time to cover the required background) that it also satisfies (F.1) and (F.2). If some model doesn’t satisfy the generic conditions at a potential bifurcation, then the potential bifurcation is not a fold bifurcation. It may be some other kind of bifurcation, or even in some pathological cases, not a bifurcation at all. If this happens, there is often some previously overlooked “exceptional” feature of the model, like a constraint, or symmetry, or other special structure (e.g. Hamiltonian, or orbitally equivalent to Hamiltonian), that explains why the generic conditions are not satisfied.

Expanding \(f(x, \alpha)\) in a Taylor series at \((x_0^0, \alpha_0)\) and using (F.0.i)–(F.2), the family (*) is

\[
\dot{x} = a(\alpha - \alpha_0) + b(x - x_0^0)^2 + O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - x_0^0| + |x - x_0^0|^3).
\]

It turns out that if \(a \neq 0\) and \(b \neq 0\), then the higher-order terms

\[
O(|\alpha - \alpha_0|^2 + |\alpha - \alpha_0||x - x_0^0| + |x - x_0^0|^3)
\]

do not qualitatively affect the local dynamics