Existence, uniqueness and smooth dependence

Let $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ have a domain that is an open subset $U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$, and consider initial value problems for a parametrized family of systems of ODEs

\[ \dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0. \]  
\[(2.1)\]

Our basic theorem on initial value problems is

**Theorem 2.1.** If $f$ is $C^p$ ($p \geq 1$) on $U$, and $(t_0, x_0, \alpha) \in U$, then each member of the family of initial value problems (2.1) has a unique solution $x(t) = \varphi(t, t_0, x_0, \alpha)$ defined for $t$ belonging to a maximal open interval of existence $J = J(t_0, x_0, \alpha) \subset \mathbb{R}$ that contains $t_0$ and in general depends on $(t_0, x_0, \alpha)$. Moreover, the solution $\varphi(t, t_0, x_0, \alpha)$ is $C^p$ in all its variables $(t, t_0, x_0, \alpha)$.

Ignoring parameter dependence for now, we let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, with a domain that is an open subset of $\mathbb{R} \times \mathbb{R}^n$, consider the initial value problem for a system of ODEs

\[ \dot{x} = f(t, x), \quad x(0) = x_0, \]  
\[(2.2)\]

and note that Theorem 2.1 can be applied to existence, uniqueness and smooth dependence of the solution $x(t) = \varphi(t, t_0, x_0)$ of (2.2). For $(t_0, x_0) \in U$, the unique solution curve containing $(t_0, x_0)$ is

\[ Cr(t_0, x_0) = \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n : x = \varphi(t, t_0, x_0), t \in J(t_0, x_0) \}. \]

**Vector fields and flows**

If $f : \mathbb{R}^n \to \mathbb{R}^n$, with a domain that is an open subset of $\mathbb{R}^n$, does not depend explicitly on $t$, then the system of ODEs $\dot{x} = f(x)$ is called **autonomous**. In this case, without loss of generality we can always take the initial time to be $t_0 = 0$,

(see Homework Assignment 2) and consider the initial value problem

\[ \dot{x} = f(x), \quad x(0) = x_0. \]  
\[(2.3)\]

In (2.3) we call $f$ the **vector field**, we call $x$-values **states**, and the domain of $f$ is the **state space**. The collection of the solutions $\varphi(t, 0, x_0)$ of all initial value problems (2.3) is called the **local flow** (generated by the vector field $f$) in $\mathbb{R}^n$. We use the notation

\[ \varphi^t(x_0) = \varphi(t, 0, x_0), \]

and call each $\varphi^t$ an **evolution operator**. If the vector field $f$ is $C^p$ on its domain, then for each fixed $t$, the evolution operator $\varphi^t$ is a local $C^p$ diffeomorphism in $\mathbb{R}^n$, with a domain that is an open subset of the domain of $f$. The evolution operators satisfy
\[ \varphi^0 = \text{id}, \quad (\text{DS.0}) \]
\[ \varphi^{s+t} = \varphi^s \circ \varphi^t \quad \text{for all } s, t \in \mathbb{R}, \quad (\text{DS.1}) \]
whenever both sides are defined, where \( \text{id} \) is the identity map in \( \mathbb{R}^n \), i.e. \( \text{id}(x) = x \) for all \( x \in \mathbb{R}^n \).

**Example 2.A.** \( \dot{x} = -x^2, \ x(0) = x_0 \in \mathbb{R}^1 \).

If \( X \) is \( \mathbb{R}^n \), or more generally, a manifold, and if \( \varphi'(x_0) \) is defined for all times \( t \in \mathbb{R} \) and all initial states \( x_0 \in X \), then the local flow is called a **global flow on X**, properties (DS.0)–(DS.1) hold everywhere on \( X \), and the triple \( \{\mathbb{R}, X, \varphi \} \) is called a **continuous-time dynamical system**. For brevity we will often call a local flow or a global flow or a continuous-time dynamical system a “flow” or “dynamical system”, and denote it by \( \varphi' \) or \( \dot{x} = f(x) \). Sometimes we will consider flows on manifolds \( X \) other than \( \mathbb{R}^n \) (examples later).

For a flow, the orbit of a state \( x_0 \) is
\[ \text{Or}(x_0) = \{x \in X : x = \varphi^t(x_0) \text{ for all } t \in J(0, x_0)\}, \]
oriented by the direction of increasing \( t \), and the phase portrait of a flow is the partitioning of the state space into orbits. Orbits \( \text{Or}(x_0) \) are projections of solution curves \( C(0, x_0) \) onto the state space (these projections are well defined for flows).

A subset \( S \subset X \) is an **invariant set**, **positively** invariant set, **negatively** invariant set, or **locally** invariant set, if \( x(0) = x_0 \in S \) implies \( x(t) = \varphi^t(x_0) \in S \) for all \( t \in J(0, x_0) = \mathbb{R} \), \( t \in J(0, x_0) \cap [0, +\infty) \) where \( [0, +\infty) \subset J(0, x_0), t \in J(0, x_0) \cap (-\infty, 0] \) where \( (-\infty, 0] \subset J(0, x_0), \) or \( t \in (-\delta_1, \delta_2) \subset J(0, x_0) \) for \( -\infty \leq -\delta_1 < 0 < \delta_2 \leq +\infty \), respectively.

An invariant set \( S_0 \) is (a) **Liapunov stable** if for any open subset \( U \subset X \) containing \( S_0 \), there exists an open subset \( V \subset X \) containing \( S_0 \), such that \( x(0) = x_0 \in V \) implies \( x(t) = \varphi^t(x_0) \in U \) for all \( t \in [0, +\infty) \) (notice we must have \( [0, +\infty) \subset J(0, x_0) \) for all \( x_0 \in V \); (b) **asymptotically stable** if there exists an open subset \( U_0 \subset X \) containing \( S_0 \) such that \( x(0) = x_0 \in U_0 \) implies \( \text{dist}(\varphi^t(x_0), S) \to 0 \) as \( t \to +\infty \); (c) **stable** if both (a) and (b) are true; (d) **unstable** if (a) is false.

An **equilibrium** is a solution \( x = p^0 \), of
\[ f(x) = 0, \]
and the set \( \{p^0\} \), consisting of an equilibrium \( p^0 \), is clearly an invariant set.

**Maps**

Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be a \( C^p \) \((p \geq 1)\) local diffeomorphism in \( \mathbb{R}^n \) with a domain that is an open subset of \( \mathbb{R}^n \), and consider the map
\[ x \mapsto f(x), \quad (2.4) \]
or equivalently the **recursion**
\[ x_{k+1} = f(x_k), \quad k \in \mathbb{Z}. \quad (2.4') \]
Then we find that
\[ x_k = f^k(x_0), \quad k \in \mathbb{Z}, \]
and the set \( \{p^0\} \), consisting of an equilibrium \( p^0 \), is clearly an invariant set.
where \( f^0 = \text{id} \) is the identity map, \( f^1 = f \), \( f^2 = f \circ f \), \( f^3 = f \circ f \circ f \), etc.; and \( f^{-1} \) is the inverse map, \( f^{-2} = f^{-1} \circ f^{-1} \), \( f^{-3} = f^{-1} \circ f^{-1} \circ f^{-1} \), etc. Each \( f^k \), \( k \in \mathbb{Z} \), is called an evolution operator. The evolution operators satisfy
\[
\begin{align*}
  f^0 &= \text{id}, \\
  f^{j+k} &= f^j \circ f^k & \text{for all } j, k \in \mathbb{Z},
\end{align*}
\]
whenever both sides are defined. A global diffeomorphism \( f \) on a manifold \( X \) generates a discrete-time dynamical system \( \{ \mathbb{Z}, X, f^k \} \) where the family of evolution operators \( \{ f^k \}_{k \in \mathbb{Z}} \) are the iterates of \( f \) or of its inverse \( f^{-1} \). If \( f \) is a local diffeomorphism but not a global diffeomorphism, then \( f^k(x_0) \) might not be defined for all \( x_0 \) or for all \( k \in \mathbb{Z} \). If \( f \) is not invertible, we can still consider \( f^k \) for nonnegative integers \( k \). For brevity we will often call a local diffeomorphism or a global diffeomorphism or a discrete-time dynamical system a “map” or a “diffeomorphism” or a “dynamical system”, and denote it by \( x \mapsto f(x) \) or \( x_{k+1} = f(x_k) \). We can consider maps in manifolds \( X \) other than \( \mathbb{R}^n \).

Orbits, phase portraits, invariant sets for maps have similar definitions as for flows.

A fixed point for a map is a solution \( x = p^0 \), of
\[
  f(x) = x,
\]
and the set \( \{ p^0 \} \) consisting of a fixed point \( p^0 \) is an invariant set.

The definitions for Lyapunov stable, asymptotically stable, stable and unstable invariant sets for maps are also similar to the corresponding definitions for flows.

Linearization and hyperbolicity

(a) Flows: If \( p^0 \) is an equilibrium of \( \dot{x} = f(x) \), then the linearization (or variational equation) at \( p^0 \) is the linear vector field
\[
  \dot{\xi} = A\xi, \quad \text{where } A = f_x(p^0).
\]
An equilibrium \( p^0 \) is hyperbolic if the linearization (2.5) is hyperbolic, i.e. if \( \Re \lambda_j \neq 0 \) for all eigenvalues \( \lambda_j \) of the constant real \( n \times n \) matrix \( f_x(p^0) \). We often call the eigenvalues of \( f_x(p_0) \) the eigenvalues of \( p^0 \).

(b) Maps: If \( p^0 \) is a fixed point of \( x \mapsto f(x) \), then the linearization at \( p^0 \) is the linear map
\[
  \xi \mapsto A\xi, \quad \text{where } A = f_x(p^0).
\]
A fixed point \( p^0 \) is hyperbolic if the linearization (2.6) is hyperbolic, i.e. \( \abs{\mu_j} \neq 1 \) for all eigenvalues \( \mu_j \) of the constant real \( n \times n \) matrix \( f_x(p^0) \). For maps, the eigenvalues of \( A = f_x(p^0) \) are called the multipliers of (2.6), and we often call them the multipliers of \( p^0 \).

Topological equivalence

To compare two different dynamical systems (both continuous-time, or both discrete-time), we would like to define when they have “qualitatively the same dynamics”. A dynamical system is topologically equivalent to a second dynamical system if there is a homeomorphism mapping the orbits of the first dynamical system onto the orbits of the second dynamical system, preserving the orientation of time.
Topological equivalence may be local or global, depending on whether the homeomorphism is local or global. In this course, mostly we are concerned with local topological equivalence.

For maps, topological equivalence can be shown to be the same as topological conjugacy: two maps \( f \) and \( g \) are topologically conjugate if there is a homeomorphism \( h \) such that \( h \circ f = g \circ h \). Topological conjugacy may be local or global.

If an equilibrium or a fixed point is hyperbolic, then the linearization gives a reliable qualitative description of the local dynamics. In fact, at a hyperbolic equilibrium or fixed point, one only needs to know the locations in the complex plane of the eigenvalues or multipliers of the linearizations relative to the imaginary axis or the unit circle, respectively, in order to determine whether two systems are locally topologically equivalent:

**Theorem 2.2.** Two local flows at hyperbolic equilibria are locally topologically equivalent if and only if their linearizations at their respective equilibria have the same dimensions \( n_\pm = \dim T^\pm \), of stable and unstable subspaces.

**Theorem 2.3.** Two maps at hyperbolic fixed points are locally topologically equivalent if and only if their linearizations at their respective fixed points have (a) the same dimensions \( n_\pm = \dim T^\pm \), of stable and unstable subspaces, and (b) the same signs of the products, of all multipliers with \( |\mu_j| < 1 \) and of all multipliers with \( |\mu_j| > 1 \).

Theorems 2.2 and 2.3 imply the Hartman-Grobman Theorem: a dynamical system at a hyperbolic equilibrium or fixed point is locally topologically equivalent to its linearization at the origin, and the Hartman-Grobman Theorem implies the Principle of Linearized Stability: the stability of a hyperbolic equilibrium or fixed point is the same as the stability of the origin for the linearization.

It is usually easier to compare vector fields directly, rather than their flows (in applications a vector field is often given explicitly as a system of differential equations, but the flow is the collection of all the solutions of the system of differential equations). Unfortunately, there is no easy general way to decide if two flows are topologically equivalent by studying only their vector fields (with two important exceptions being linear flows, and nonlinear local flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice:

(a) If a vector field \( f \) is smoothly equivalent (or \( C^p \) equivalent or diffeomorphic or \( C^p \) diffeomorphic) to a second vector field \( g \):

\[
f(x) = M(x)^{-1} g(h(x)),
\]

where \( M(x) = h_x(x) \), for some \( C^p \) (\( p \geq 1 \)) diffeomorphism \( h \) (i.e. a smooth change of state space variables \( y = h(x) \) transforms \( \frac{dy}{dt} = g(y) \) into \( \frac{dx}{dt} = f(x) \)), then the two corresponding flows are topologically equivalent.

(b) If a vector field is orbitally equivalent to a second vector field \( g \):

\[
f(x) = \mu(x)g(x)
\]

for some smooth real-valued, strictly positive function \( \mu : \mathbb{R}^n \to (0, \infty) \) (i.e. a smooth rescaling of the time variable \( s = \mu(x)t \) transforms \( \frac{dx}{ds} = g(x) \) into \( \frac{dx}{dt} = f(x) \)), then the two corresponding flows are topologically equivalent.

Of course, (a) and (b) can be combined.
Linearization at cycles

(a) Flows: If \( L_0 = \{ p^0(t) \}_{t \in \mathbb{R}} \) is a cycle (or periodic orbit) of period \( T_0 > 0 \) (a \( T_0 \)-cycle) for the flow generated by the vector field \( \dot{x} = f(x) \), then the variational equation (or linearization) of the vector field at \( p^0(t) \) is

\[
\dot{u} = A(t)u, \quad \text{where } A(t) = f_x(p^0(t)).
\]

Since \( A(t) = f_x(p^0(t)) \) is a continuous periodic real \( n \times n \) matrix with period \( T_0 \), we can use Floquet theory. If \( L_0 \) is a \( T_0 \)-cycle for a flow, then one of the Floquet multipliers of the linearization at \( p^0(t) \) (2.7) must be equal to 1 (Exercise), so the Floquet multipliers, counting multiplicities, are

\[ \mu_1, \ldots, \mu_{n-1}, 1 \]

(not necessarily all distinct). The \( n - 1 \) nontrivial Floquet multipliers \( \mu_1, \ldots, \mu_{n-1} \) contain linearized stability information for the cycle. The cycle is hyperbolic if none of the nontrivial Floquet multipliers is on the unit circle \(|\mu|=1\).

A cycle \( L_0 \) for a flow is a limit cycle if there is an open set in the state space that contains \( L_0 \) but no other cycles. (If a cycle is hyperbolic, then it must be a limit cycle.)

Example 2.B. A hyperbolic cycle for

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) = x_1 - \omega x_2 - x_1^3 - x_1x_2^2, \\
\dot{x}_2 &= f_2(x_1, x_2) = \omega x_1 + x_2 - x_1^2x_2 - x_2^3.
\end{align*}
\]

(b) Maps: For a map \( x \mapsto f(x) \), \( L_0 = \{ p^0_0, \ldots, p^0_{K_0-1} \} \) is a \( K_0 \)-cycle (for maps, the period \( K_0 \) must be a positive integer) if and only if each point \( p^0_j \) in the cycle \( L_0 \) is a fixed point of the \( K_0 \)th iterate map \( x \mapsto f^{K_0}(x) \). To study the stability of a \( K_0 \)-cycle for \( x \mapsto f(x) \), we linearize the \( K_0 \)th iterate map at any one of its \( K_0 \) fixed points \( p^0_j \). The cycle \( L_0 \) is hyperbolic if one of the points in the \( K_0 \)-cycle for the map \( x \mapsto f(x) \) is a hyperbolic fixed point for the \( K_0 \)th iterate map \( x \mapsto f^{K_0}(x) \) (and of course, you could prove that it does not matter which point in the cycle is chosen).

Poincaré maps

Near a \( T_0 \)-cycle \( L_0 = \{ p^0(t) \} \) for a flow \( \varphi^t \) generated by a vector field \( f \), we can study the fully nonlinear (but perhaps only local) dynamics by means of a Poincaré map, defined in a smooth \((n-1)\)-dimensional cross-section \( \Sigma \) at a point \( p^0_0 = p^0(t_0) \) on the cycle. The Poincaré map is a local diffeomorphism in \( \Sigma \), obtained by following the solution \( x(t) = \varphi^t(x_0) \) of the initial value problem for \( \dot{x} = f(x) \) starting from a state \( x(0) = x_0 \) in \( \Sigma \) sufficiently near \( p^0_0 \), until the first time \( T(x_0) > 0 \) (near the period \( T_0 \) of the cycle) that the solution returns to \( \Sigma \), then defining \( P : \Sigma \to \Sigma \) by \( P(x_0) = \varphi^{T(x_0)}(x_0) \). Therefore, \( p^0_0 \) is a fixed point of \( P \). If the vector field \( f \) is \( C^p \), then the Poincaré map \( P \) is a local \( C^p \) diffeomorphism.

To make explicit calculations, it is convenient to define some smooth local coordinates \( \xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \) on \( \Sigma \). Arranging for \( \xi = 0 \in \mathbb{R}^{n-1} \) to correspond to \( x = p^0_0 \in \Sigma \), we can express the Poincaré map in coordinate form as \( \xi \mapsto P(\xi) \), where \( P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \), with a fixed point at \( 0 \in \mathbb{R}^{n-1} \).

The stability of a cycle \( \{ p^0(t) \} \) for the flow corresponds to the stability of the fixed point 0 for the Poincaré map \( P \) (in its coordinate version). Linearized stability of the cycle is determined by
the multipliers (eigenvalues) $\mu_1, \ldots, \mu_{n-1}$ of the linearization of the Poincaré map at its fixed point 0, the $(n - 1) \times (n - 1)$ matrix $P_\xi(0)$. These multipliers can be shown to be independent of the choice of the point $p^0_0$ on the cycle, of the choice of the cross-section $\Sigma$ at $p^0_0$, and of the choice of the coordinates $\xi$ on $\Sigma$. Moreover, we have the following:

**Theorem 2.4.** The nontrivial Floquet multipliers of the linearization $\dot{u} = f_\xi(p^0(t))u$, of $\dot{x} = f(x)$ at the cycle $\{p^0(t)\}_{t \in \mathbb{R}}$, are the same as the multipliers of the linearization $\eta \mapsto P_\xi(0)\eta$, of the Poincaré map $\xi \mapsto P(\xi)$ at its corresponding fixed point 0 in $\mathbb{R}^{n-1}$.

Thus, a cycle for a flow is hyperbolic if and only if the corresponding fixed point for a Poincaré map is hyperbolic.

**Example 2.C.** A Poincaré map for the cycle in Example 2.B.