Existence, uniqueness, smooth dependence

Let \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) be \( C^p \) (\( p \geq 1 \)) on its domain, an open set \( U \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \), and consider initial value problems for a parametrized family of ODEs

\[
\dot{x} = f(t,x,\alpha), \quad x(t_0) = x_0 \in \mathbb{R}^n.
\]  

(2.1)

Our basic theorem on initial value problems is

**Theorem 2.1.** If \((t_0,x_0,\alpha) \in U\), then each member of the family of initial value problems (2.1) has a unique solution \( x(t) = \varphi(t,t_0,x_0,\alpha) \) defined for \( t \) belonging to a maximal open interval of existence \( J = J(t_0,x_0,\alpha) \subseteq \mathbb{R} \) that in general depends on \((t_0,x_0,\alpha)\). The solution \( \varphi(t,t_0,x_0,\alpha) \) is \( C^p \) in all variables \((t,t_0,x_0,\alpha)\).

Ignoring parameter dependence for now, we let \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \), consider the initial value problem for a single ODE

\[
\dot{x} = f(t,x), \quad x(0) = x_0 \in \mathbb{R}^n,
\]  

(2.2)

and note that Theorem 2.1 can be applied to the solution \( x(t) = \varphi(t,t_0,x_0) \) of (2.2). We define for \((t_0,x_0)\) belonging to the domain of \( f \) in \( \mathbb{R} \times \mathbb{R}^n \), the solution curve passing through \((t_0,x_0)\),

\[
Cr(t_0,x_0) = \{(t,x) : x = \varphi(t,t_0,x_0), \ t \in J(t_0,x_0)\} \subseteq \mathbb{R} \times \mathbb{R}^n.
\]

Autonomous ODEs, local flows

If \( f : \mathbb{R}^n \to \mathbb{R}^n \) does not depend explicitly on \( t \), then \( \dot{x} = f(x) \) is called an autonomous ODE. In this case, without loss of generality we can always take the initial time

\[
t_0 = 0,
\]

and consider the initial value problem

\[
\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R}^n.
\]  

(2.3)

We call \( f(x) \) the vector field, we call \( x \)-values states, and the domain \( U \subseteq \mathbb{R}^n \) of \( f \) is the state space. The collection of the solutions \( \varphi(t,0,x_0) \) of all initial value problems (2.3) is the local flow generated by \( f \), we use the notation

\[
\varphi^t(x_0) = \varphi(t,0,x_0),
\]

and call \( \varphi^t \) the evolution operator. If the vector field is \( C^p \) then, for each fixed \( t \), the evolution operator \( \varphi^t : \mathbb{R}^n \to \mathbb{R}^n \) is a local \( C^p \) diffeomorphism. The evolution operators satisfy

\[
\varphi^0 = \text{id}, \quad (D.0)
\]

where \( \text{id} \) is the identity map in \( \mathbb{R}^n \), and

\[
\varphi^{t+s} = \varphi^t \circ \varphi^s \quad \text{for all} \ t, s \in \mathbb{R} \ \text{whenever both sides are defined}. \quad (D.1)
\]

If \( X \) is \( \mathbb{R}^n \) or a manifold, and if \( \varphi^t(x_0) \) is defined for all \( t \in \mathbb{R} \) and all initial states \( x_0 \in X \), then the local flow \( \{\varphi^t\}_{t \in \mathbb{R}} \) is a (global) flow on \( X \), and the triple \( \{\mathbb{R}, X, \varphi^t\} \) is a continuous-time dynamical
system. But for brevity we often say flow or dynamical system, when we actually are referring to a local flow. Sometimes we consider flows on manifolds other than $\mathbb{R}^n$.

For an autonomous ODE, the orbit of a state $x_0$ is

$$\text{Or}(x_0) = \{ x \in \mathbb{R}^n : x = \varphi^t(x_0) \text{ for all } t \in J(0, x_0) \},$$
oriented by the direction of increasing $t$, and the **phase portrait** of a flow is the partitioning of the state space into orbits. Orbits are projections of solution curves onto the state space (the projections are well defined if the ODE is autonomous).

A set $S \subseteq \mathbb{R}^n$ is an **invariant** set, **positively** invariant set, **negatively** invariant set, or **locally** invariant set, if $x(0) = x_0 \in S$ implies $x(t) = \varphi^t(x_0) \in S$ for all $t \in J(0, x_0)$, $t \in J(0, x_0) \cap [0, +\infty)$, $t \in J(0, x_0) \cap (-\infty, 0]$, or $t \in (-\delta_1, \delta_2) \subseteq J(0, x_0)$ for $-\infty \leq -\delta_1 < 0 < \delta_2 \leq +\infty$, respectively.

An invariant set $S_0$ is (i) **Liapunov stable** if for any sufficiently small open set $U$ containing $S_0$, there exists an open set $V$ containing $S_0$ such that $x(0) = x_0 \in V$ implies $x(t) = \varphi^t(x_0) \in U$ for all $t \in [0, +\infty)$ (notice we must have $[0, +\infty) \subseteq J(0, x_0)$ for all $x_0 \in V$); (ii) **asymptotically stable** if there exists an open set $U_0$ containing $S_0$ such that $x(0) = x_0 \in U_0$ implies $x(t) = \varphi^t(x_0) \to S_0$ (i.e. dist$(\varphi^t(x_0), S_0) \to 0)$ as $t \to +\infty$; (iii) **stable** if both (i) and (ii) are true; (iv) **unstable** if (i) is false.

An **equilibrium** is a solution of

$$f(x) = 0,$$
and the set consisting of an equilibrium is clearly an invariant set.

Maps

For $f : \mathbb{R}^n \to \mathbb{R}^n$ we consider maps

$$x \mapsto f(x), \ x \in \mathbb{R}^n, \tag{2.4}$$
or equivalently

$$x_{k+1} = f(x_k), \ x_k \in \mathbb{R}^n, \ k \in \mathbb{Z},$$
assuming that $f$ is a local diffeomorphism in $\mathbb{R}^n$. Then

$$x_k = f^k(x_0), \ k \in \mathbb{Z},$$
and each $f^k$ is an **evolution operator**. The evolution operators satisfy

$$f^0 = \text{id}, \tag{D.0}$$
where $\text{id}$ is the identity map in $\mathbb{R}^n$, and

$$f^{k+j} = f^k \circ f^j \text{ for all } k, j \in \mathbb{Z} \text{ whenever both sides are defined.} \tag{D.1}$$
A global diffeomorphism $f$ on a manifold $X$ generates a discrete-time dynamical system $\{Z, X, f^k\}$ where the family of evolution operators $\{f^k\}_{k \in \mathbb{Z}}$ are the iterates of $f$ or of its inverse $f^{-1}$. If $f$ is not a global diffeomorphism, then $f^k(x_0)$ might not be defined for all $k \in \mathbb{Z}$ or for all $x_0$, but we will often still call it a map, or a diffeomorphism, or a discrete-time system.

**Orbits, phase portraits, invariant** sets for maps have similar definitions as for flows. The definitions for **Lyapunov stable**, **asymptotically stable**, **stable** and **unstable** invariant sets for maps are also similar to the corresponding definitions for flows.

A **fixed point** for a map is a solution of

$$f(x) = x,$$
and the set consisting of a fixed point is an invariant set.
Linearization, hyperbolicity

(a) Flows: If \( x^0 \) is an equilibrium of \( \dot{x} = f(x) \), then the linearization (or variational equation) at \( x^0 \) is the linear vector field
\[
\dot{\xi} = A\xi, \text{ where } A = f_x(x^0).
\]  
(2.5)

An equilibrium \( x^0 \) is hyperbolic if the linearization (2.5) is hyperbolic, i.e. \( \Re \lambda_j \neq 0 \) for all eigenvalues of the linearization.

(b) Maps: If \( x^0 \) is a fixed point of \( x \mapsto f(x) \), then the linearization at \( x^0 \) is the linear map
\[
\xi \mapsto A\xi, \text{ where } A = f_x(x^0).
\]  
(2.6)

For maps, the eigenvalues of \( A \) are called the multipliers of \( x^0 \). A fixed point \( x^0 \) is hyperbolic if the linearization, i.e. (2.6) is hyperbolic i.e. \( |\mu_j| \neq 1 \) for all multipliers of the linearization.

Topological equivalence

To compare two different dynamical systems, we would like to be able to decide when they have “qualitatively the same dynamics”. A (continuous-time or discrete-time) dynamical system is topologically equivalent to a second dynamical system if there is a homeomorphism mapping the orbits of the first system onto the orbits of the second system, preserving the direction of time.

Topological equivalence may be local or global.

For maps, topological equivalence can be shown to be the same as topological conjugacy (two maps \( f \) and \( g \) are topologically conjugate if there is a homeomorphism \( h \) such that \( h \circ f = g \circ h \)).

If an equilibrium or a fixed point is hyperbolic, it turns out that linearization gives a reliable qualitative description of the local dynamics. At a hyperbolic equilibrium or fixed point, one only needs to know the locations in the complex plane of the eigenvalues or multipliers of the linearizations relative to the imaginary axis or unit circle, respectively, in order to determine whether two systems are locally topologically equivalent.

**Theorem 2.2.** Two local flows at hyperbolic equilibria, \( x^0 \) for \( \frac{dx}{dt} = f(x) \) and \( y^0 \) for \( \frac{dy}{ds} = g(y) \), are locally topologically equivalent if and only if their linearizations at their respective equilibria have the same the dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \) of stable and unstable subspaces.

**Theorem 2.3.** Two maps at hyperbolic fixed points, \( x^0 \) for \( x \mapsto f(x) \) and \( y^0 \) for \( y \mapsto g(y) \), are locally topologically equivalent if and only if their linearizations at their respective fixed points have i) the same dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \) of stable and unstable subspaces, and ii) the same signs of products of all multipliers with \( |\mu_j| < 1 \) and with \( |\mu_j| > 1 \).

Theorems 2.2 and 2.3 imply the principle of linear(ized) stability: the stability of a hyperbolic equilibrium or fixed point is the same as the stability of the origin for the linearization.

It is usually easier to compare vector fields rather than their flows (in applications a vector field is often given explicitly as a system of differential equations but the flow is essentially all the solutions of the system of differential equations). Unfortunately, there is no easy general way to decide if two flows are topologically equivalent by studying only their vector fields (but two important exceptions are linear flows, and nonlinear flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice:

(i) If a vector field \( f \) is smoothly equivalent or diffeomorphic to a second vector field \( g \),
\[
f(x) = h_x(x)^{-1}g(h(x))
\]
for some diffeomorphism \( h \) (i.e. \( \frac{dx}{dt} = f(x) \) is transformed into \( \frac{dy}{dt} = g(y) \) by a smooth change of state space variables \( y = h(x) \)), then the two corresponding flows are topologically equivalent.

(ii) If a vector field is \textbf{orbitally equivalent} to a second vector field \( g \),

\[ f(x) = \mu(x)g(x) \]

for some smooth \textit{scalar, positive} function \( \mu : \mathbb{R}^n \to (0, \infty) \) (i.e. \( \frac{dx}{dt} = f(x) \) is transformed into \( \frac{dx}{ds} = g(x) \) by a smooth state space-dependent rescaling of the time variable), then the two corresponding flows are topologically equivalent.

Of course, (i) and (ii) can be combined.

\textbf{Linearization at cycles}

(a) \textit{Flows:} If \( L_0 = \{x^0(t)\} \) is a \textit{cycle} (or \textit{periodic orbit}) of period \( T_0 > 0 \) (a \( T_0 \)-\textit{cycle}) for the flow generated by the vector field \( \dot{x} = f(x) \), then the \textit{variational equation} (or \textit{linearization}) of the vector field at \( x^0(t) \) is

\[ \dot{u} = A(t)u, \quad \text{where } A(t) = f_x(x^0(t)). \] (2.7)

Since \( A(t) = f_x(x^0(t)) \) is a continuous periodic \( n \times n \) matrix with period \( T_0 \), we can use Floquet theory. If \( \{x^0(t)\} \) is a \( T_0 \)-cycle for an autonomous ODE, then one of the Floquet multipliers of the linearization at \( x^0(t) \) must be equal to 1, so the Floquet multipliers are

\[ \mu_1, \ldots, \mu_{n-1}, 1 \]

(not necessarily all distinct). The \( n - 1 \) \textit{nontrivial} Floquet multipliers \( \mu_1, \ldots, \mu_{n-1} \) contain linearized stability information. The cycle is \textit{hyperbolic} if none of the nontrivial Floquet multipliers is on the unit circle \(|\mu| = 1\).

A cycle \( L_0 \) for a flow in \( \mathbb{R}^n \) is a \textit{limit cycle} if there is an open set in \( \mathbb{R}^n \) that contains \( L_0 \) but no other cycles. If a cycle is hyperbolic then it must be a limit cycle.

(b) \textit{Maps:} For a map \( x \mapsto f(x) \), a set of discrete points \( L_0 = \{x^0_0, \ldots, x^0_{K_0-1}\} \) is a \( T_0 \)-cycle if and only if each point \( x^0_j \) in the cycle is a fixed point of the \( T_0 \)-iterate \( f^{T_0} \). To study the stability of a \( T_0 \)-cycle for \( x \mapsto f(x) \), we linearize the \( T_0 \)-iterate map \( x \mapsto f^{T_0}(x) \) at any one of its \( T_0 \) fixed points. The cycle is \textit{hyperbolic} if one of the points \( x^0_j \) is a hyperbolic fixed point for \( f^{T_0} \) (of course, for this definition it can be proved that it does not matter which point in the cycle is chosen).

\textbf{Poincaré maps}

Near a \( T_0 \)-cycle \( L_0 = \{x^0(t)\} \) for a flow generated by an autonomous ODE \( \dot{x} = f(x) \), we can study the fully nonlinear (but perhaps only local) dynamics by means of a \textbf{Poincaré map}, defined in a smooth \((n - 1)\)-dimensional \textbf{cross-section} \( \Sigma \) at a point \( x^0_0 = x^0(t_0) \) on the cycle. The Poincaré map is a local diffeomorphism in \( \Sigma \), obtained by following the solution to the ODE \( x(t) = \varphi^t(x_0) \) starting from any state \( x(0) = x_0 \) in \( \Sigma \), sufficiently near \( x^0_0 \), until the first time \( T(x_0) > 0 \) (near the period \( T_0 \) of the cycle) that the solution returns to \( \Sigma \), defining \( P : \Sigma \to \Sigma \) by \( P(x_0) = \varphi^{T(x_0)}(x_0) \). Therefore, \( x^0_0 \) is a fixed point of the Poincaré map \( P \).

Using nonsingular, smooth local coordinates \( \xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \) on \( \Sigma \) (arranging for \( \xi = 0 \in \mathbb{R}^{n-1} \) to correspond to \( x = x^0_0 \in \Sigma \)), we can express the Poincaré map in coordinate form as \( \xi \mapsto P(\xi) \), where \( P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \), with a fixed point at \( 0 \in \mathbb{R}^{n-1} \). This makes explicit calculations possible.

The stability of the cycle \( \{x^0(t)\} \) for the flow corresponds to the stability of the fixed point \( 0 \) for the Poincaré map \( P \) (in its \( \xi \)-coordinate version). Linearized stability of the cycle is determined by the multipliers (eigenvalues) \( \mu_1, \ldots, \mu_{n-1} \) of the linearization of the Poincaré map at its fixed point \( 0 \), the \((n - 1) \times (n - 1)\) matrix \( A = P'_0(0) \). These multipliers can be shown to be independent of the choice of the point \( x^0_0 \) on the cycle, of the choice of the cross-section \( \Sigma \) at \( x^0_0 \), and of the choice of the coordinates \( \xi \) on \( \Sigma \). Moreover, we have the following theorem.
**Theorem 2.4.** The nontrivial Floquet multipliers of the linearization \( \dot{u} = f(x) \) of the autonomous ODE \( \dot{x} = f(x) \) at the cycle \( \{x^0(t)\} \) in \( \mathbb{R}^n \), are the same as the multipliers of the linearization \( v \mapsto \Phi_t(0)v \), of the Poincaré map \( \xi \mapsto \Phi(\xi) \) at its corresponding fixed point \( 0 \) in \( \mathbb{R}^{n-1} \).

Thus, a cycle for a flow is hyperbolic if and only if the corresponding fixed point for a Poincaré map is hyperbolic.

**Local and global invariant manifolds**

If \( x^0 \) is an equilibrium of a flow or a fixed point of a map, then its stable set \( W^s(x^0) \) is the set of all states whose orbits approach \( x^0 \) in forward time, while its unstable set \( W^u(x^0) \) is the set of all states whose orbits approach \( x^0 \) in backward time. Both these sets are invariant. If \( x^0 \) is hyperbolic, we have further results.

**Theorem 2.5** (Local stable and unstable manifolds for flows). If \( f \) is \( C^p \) \((p \geq 1)\) and \( x^0 \) is a hyperbolic equilibrium for \( \dot{x} = f(x) \), then the intersections of \( W^s(x^0) \) and \( W^u(x^0) \) with a sufficiently small open neighbourhood of \( x^0 \) contain \( C^p \) submanifolds \( W^s_{loc}(x^0) \) and \( W^u_{loc}(x^0) \) of dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \), respectively. The smooth submanifolds \( W^s_{loc}(x^0) \) and \( W^u_{loc}(x^0) \) are tangent at \( x^0 \) to \( T^s \) and \( T^u \), respectively, the stable and unstable subspaces of the linearization at \( x^0 \).

**Theorem 2.6** (Local stable and unstable manifolds for maps). If \( f \) is \( C^p \) \((p \geq 1)\) and \( x^0 \) is a hyperbolic fixed point for \( x \mapsto f(x) \), then the intersections of \( W^s(x^0) \) and \( W^u(x^0) \) with a sufficiently small open neighbourhood of \( x^0 \) contain \( C^p \) submanifolds \( W^s_{loc}(x^0) \) and \( W^u_{loc}(x^0) \) of dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \), respectively. The smooth submanifolds \( W^s_{loc}(x^0) \) and \( W^u_{loc}(x^0) \) are tangent at \( x^0 \) to \( T^s \) and \( T^u \), respectively, the stable and unstable subspaces of the linearization at \( x^0 \).

If the equilibrium or fixed point \( x^0 \) is hyperbolic, then letting all states in the positively invariant local stable manifold \( W^s_{loc}(x^0) \) evolve backwards in time we recover \( W^s(x^0) \), and similarly letting all states in the negatively invariant local unstable manifold \( W^u_{loc}(x^0) \) evolve forwards in time we recover \( W^u(x^0) \). This implies that \( W^s(x^0) \) and \( W^u(x^0) \) are locally \( C^p \) submanifolds. For this reason (if \( x^0 \) is hyperbolic) the stable and unstable sets \( W^s(x^0) \) and \( W^u(x^0) \) are often referred to as the (global) stable and unstable manifolds of \( x^0 \), respectively. They are important features of the dynamics. Sometimes these global invariant manifolds have complicated features.

**Hamiltonian systems**

A Hamiltonian system is a vector field in the even-dimensional state space \( \mathbb{R}^{2s} \), generated by a real-valued \( C^{p+1} \) (\( p \geq 1 \)) Hamiltonian function \( H \) with domain in \( \mathbb{R}^{2s} \), of the form

\[ \dot{q} = H_y(q,p), \quad \dot{p} = -H_x(q,p), \]

where \( x = (q,p) \in \mathbb{R}^s \times \mathbb{R}^s = \mathbb{R}^{2s} \). In a Hamiltonian system, all solutions \( x(t) = (q(t),p(t)) \) satisfy \( (d/\,dt)H(x(t)) = 0 \) and therefore \( x(t) \) remain on level sets \( H(q,p) = \text{constant} \). This fact ("conservation of energy") makes determining the global phase portrait especially easy in the case of \( s = 1 \).

**Lyapunov functions**

A Lyapunov function \( L(x) \) for an autonomous ODE \( \dot{x} = f(x) \) is a smooth function \( L : \mathbb{R}^n \to \mathbb{R} \) with

\[ \hat{L}(x) \leq 0 \]

for all \( x \) in some open region of interest, where

\[ \hat{L}(x) = \langle L_x(x), f(x) \rangle, \]
\langle \cdot, \cdot \rangle \text{ denotes the inner product (dot product) in } \mathbb{R}^n, \text{ and } L_x(x) = \nabla L(x) \text{ is the gradient of } L.

\text{Locally, a Liapunov function can be used to study stability of equilibria: if } x^0 \text{ is an equilibrium for } \dot{x} = f(x) \text{ and } L \text{ is defined and smooth in some sufficiently small open neighbourhood } U \text{ of } x^0 \text{ with}

i. \ x_0 \text{ is at an isolated local minimum of } L \text{ in } U;

ii. \ \dot{L} \leq 0 \text{ in } U;

then } x^0 \text{ is a Lyapunov stable equilibrium for } \dot{x} = f(x). \text{ If, in addition to i. and ii. we have}

iii. \ \dot{L} < 0 \text{ in } U \setminus \{x^0\};

then } x^0 \text{ is a stable (i.e. both Lyapunov stable and asymptotically stable) equilibrium.}

\text{Globally, if } \dot{L} \leq 0 \text{ (or even better, } \dot{L} < 0) \text{ in some large region of state space, a Lyapunov function can help in determining topological features of the global flow. It can also be useful in constructing a trapping region (a compact, positively invariant set with nonempty interior) provided that sublevel sets } \{x \in \mathbb{R}^n : L(x) \leq c\} \text{ for constants } c \text{ can be found with the right properties.}