1. **Homoclinic bifurcation for two-dimensional flows (Andronov-Leontovich theorem)**

Homoclinic orbits are often associated with other dynamics of interest. We summarize a bifurcation analysis near a homoclinic orbit for a family of vector fields on the plane. We consider a one-parameter family of two-dimensional vector fields

\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1, \tag{4.1}
\]

and assume

\[
f(p_0^0, \alpha^0) = 0, \quad A_0 = f_x(p_0^0, \alpha^0) \text{ has eigenvalues } \lambda_1^0 < 0 < \lambda_2^0, \tag{HC 1}
\]

and

\[
\text{for } \alpha = \alpha^0, \text{ (4.1) has an orbit } \Gamma = \{x^0(t)\} \text{ homoclinic to } p_0^0. \tag{HC 2}
\]

Thus, for \( \alpha = \alpha^0 \), (4.1) has a hyperbolic saddle equilibrium \( p_0^0 \), \( \Gamma \) is a homoclinic orbit (or saddle loop), and \( \lim_{t \to \pm \infty} x^0(t) = p_0^0 \). We would like to determine important features of the dynamics for (4.1) when \( \alpha \neq \alpha^0 \). The Implicit Function Theorem can be used to solve \( f(x, \alpha) = 0 \), to obtain a unique locally defined smooth solution \( x = p^0(\alpha) \) near \( p_0^0 \), with \( p^0(\alpha^0) = p_0^0 \), giving a curve \( (p^0(\alpha), \alpha) \) of equilibria through the point \( (p_0^0, \alpha^0) \) in \( \mathbb{R}^2 \times \mathbb{R}^1 \). Since the eigenvalues of \( A_0 \) are simple, the matrix of the linearization \( A(\alpha) = f_x(p^0(\alpha), \alpha) \) has real eigenvalues \( \lambda_1(\alpha) < 0 < \lambda_2(\alpha) \) that depend smoothly on \( \alpha \) near \( \alpha_0 \), with \( \lambda_j(\alpha^0) = \lambda_j^0, \ j = 1, 2 \), and thus \( p^0(\alpha) \) remains a hyperbolic saddle equilibrium.

Assumption (HC 2) can be expressed equivalently as

\[
\beta(\alpha_0) = 0, \tag{HC 2'}
\]

where \( \beta(\alpha) \) is a split function that measures the signed arc length along a local (one-dimensional) cross-section \( \Sigma \) to the flow, between the intersections of global stable and unstable manifolds of \( p^0(\alpha) \) with \( \Sigma \). We assume that the homoclinic orbit for \( \alpha = \alpha^0 \) “splits” (or “breaks”), in a transverse manner, as \( \alpha \) increases past \( \alpha^0 \), i.e.

\[
a = \beta'(\alpha^0) \neq 0, \tag{HC 3}
\]

so that \( \beta(\alpha) = a(\alpha - \alpha^0) + O(|\alpha - \alpha^0|^2) \), which implies that \( \beta(\alpha) \neq 0 \) for all \( \alpha \neq \alpha^0, \alpha \) near \( \alpha^0 \).

As usual, we make a coordinate shift and then a linear coordinate change so that in the new coordinates, the equilibrium is the origin for all \( \alpha \) near \( \alpha^0 \), and the local stable and unstable manifolds of the equilibrium at the origin are tangent to the coordinate axes. Further smooth global nonlinear coordinate changes make the local stable and unstable manifolds **coincide** with the coordinate axes along open segments containing the origin, and then finally **linearize** the family of vector fields in an open neighbourhood \( \Omega \) of the origin for all \( \alpha \) near \( \alpha_0 \). Thus we obtain global coordinates \( (\xi, \eta) \in \mathbb{R}^2 \), smoothly related to the original coordinates \( x = (x_1, x_2) \), so that the family of vector fields in \( \Omega \) is given in these coordinates by the decoupled linear system

\[
\begin{align*}
\dot{\xi} &= \lambda_1(\alpha)\xi, \\
\dot{\eta} &= \lambda_2(\alpha)\eta, \quad (\xi, \eta) \in \Omega \subset \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1,
\end{align*}
\]

for all \( \alpha \) near \( \alpha_0 \).

For convenience, we scale \( (\xi, \eta) \) (if necessary) so that the closed rectangle \([-1, 1] \times [-1, 1]\) lies in the open region \( \Omega \). We choose a cross-section \( \Sigma = \{(\xi, \eta) \in \Omega : \xi = 1, -1 < \eta < 1\} \) and consider the one-parameter family of Poincaré maps \( P(\cdot, \alpha) : \Sigma^+ \to \Sigma \), where \( \Sigma^+ \) is the portion of \( \Sigma \) with \( \eta > 0 \). Defining
another cross-section \( \Pi = \{(\xi, \eta) \in \Omega : -1 < \xi < 1, \eta = 1\} \), we express \( P \) as the composition of two parts \( P = Q \circ \Delta \), where the first part \( \Delta : \Sigma^+ \to \Pi \) is the result of the flow within the region \( \Omega \) where the system has been linearized, and the second part \( Q : \Pi \to \Sigma \) is the result of the flow near \( \Gamma \). An explicit expression for \( \Delta \) is easily obtained, while for the flow near \( \Gamma \) we approximate \( Q \) by its truncated Taylor series at first order. Composing the two parts, we obtain the leading-order terms

\[
P(\eta, \alpha) = a(\alpha - \alpha^0) + b\eta - \lambda_1^0/\lambda_2^0 + \cdots,
\]

where \( a \neq 0 \) and \( b > 0 \). Assume that the trace of the matrix of the linearization at the saddle equilibrium does not vanish at the critical parameter value, i.e.

\[
\sigma^0 = \text{div} f(p_0^0, \alpha^0) = \lambda_1^0 + \lambda_2^0 \neq 0.
\]

Then the Andronov-Leontovich Theorem states that the leading-order terms that we have obtained above are enough to determine the dynamics of the family of Poincaré maps \( P(\eta, \alpha) \), up to local conjugacy.

**Theorem 4.1. (Andronov-Leontovich)** If \( f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \) is \( C^2 \) in an open set containing \( \Gamma \cup \{p_0^0\} \) and satisfies conditions (HC 1)–(HC 4), then (4.1) has a family of Poincaré maps that is locally conjugate to

\[
\eta \mapsto \beta + b\eta - \lambda_1^0/\lambda_2^0 \quad \text{at} \quad (0, 0),
\]

where \( b \) is a positive constant. In particular, for all \( \alpha \) sufficiently near \( \alpha^0 \) there is an open neighbourhood \( U \) of \( \Gamma \cup \{p_0^0\} \) in \( \mathbb{R}^2 \), in which a unique limit cycle for (4.1) bifurcates from \( \Gamma \cup \{p_0^0\} \) for \( \alpha \) on only one side of \( \alpha^0 \). If \( \sigma^0 < 0 \), then the cycle is asymptotically stable and exists for \( \beta > 0 \), while if \( \sigma^0 > 0 \), then the cycle is unstable and exists for \( \beta < 0 \).

The split function is \( \beta = \beta(\alpha) = a(\alpha - \alpha^0) + O(\alpha^2 - \alpha^0) \). As \( \beta \to 0 \) from the appropriate side of 0, the limit cycle approaches \( \Gamma \cup \{p_0^0\} \), and the period of the cycle approaches infinity.

2. **Melnikov’s method**

Melnikov’s method is a global perturbation method that allows us to prove the existence or nonexistence of homoclinic solutions by perturbing from a case where a homoclinic solution is already known to exist.

For a two-dimensional Hamiltonian vector field (the “unperturbed” system) generated by a smooth Hamiltonian function \( H(x) \),

\[
\dot{x} = f(x), \quad x \in \mathbb{R}^2,
\]

where

\[
f = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right), \quad f_1(x) = H_{x_2}(x_1, x_2), \quad f_2(x) = -H_{x_1}(x_1, x_2),
\]

it is relatively easy to find homoclinic orbits. Let us assume

\[
equation (4.2) has a hyperbolic saddle equilibrium \( p_0^0 \), and an orbit \( \Gamma = \{x^0(t)\} \) homoclinic to \( p_0^0 \), \( \lim_{t \to \pm\infty} x^0(t) = p_0^0 \).
\]

Now we consider the “perturbed” one-parameter family of nonautonomous two-dimensional equations

\[
\dot{x} = F(x, t, \alpha) = f(x) + \alpha g(x, \omega t), \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}^1,
\]

where \( \omega > 0 \) is fixed, \( \alpha \) is a parameter near 0, and \( F \) is periodic in \( t \) with period \( T = 2\pi/\omega > 0 \) (\( g \) is periodic in \( \omega t \) with period \( 2\pi \)). Defining a new variable \( \theta = \omega t \), we write the family of nonautonomous two-dimensional equations (4.3) as a family of three-dimensional vector fields

\[
\dot{x} = f(x) + \alpha g(x, \theta), \quad \dot{\theta} = \omega, \quad \dot{x} = (x, \theta) \in X = \mathbb{R}^2 \times S^1, \quad \alpha \in \mathbb{R}^1.
\]

\[\]
Defining a global cross-section for (4.4) for any $\alpha$,
\[
\Sigma = \{ \dot{x} = (x, \theta) \in X : x \in \mathbb{R}^2, \theta = 0 \text{ (mod } 2\pi) \},
\]
we consider the one-parameter family of two-dimensional Poincaré maps for (4.4), $P(\cdot, \alpha) : \Sigma \to \Sigma$.

For $\alpha = 0$, the Poincaré map $P(\cdot, 0)$ is just the time-$(2\pi/\omega)$ map for the flow of the unperturbed system (4.2), so it has a hyperbolic saddle fixed point $(p_0^0, 0 \text{ (mod } 2\pi)) \in \Sigma$, whose stable and unstable manifolds coincide along $\Gamma \times \{0 \text{ (mod } 2\pi)\}$, in $\Sigma$. The hyperbolic saddle fixed point $(p_0^0, 0 \text{ (mod } 2\pi))$ for $P(\cdot, 0)$ in $\Sigma$ corresponds to a $(2\pi/\omega)$-periodic hyperbolic limit cycle $\tilde{p}_0^0(t) = (p_0^0, \omega t \text{ (mod } 2\pi))$ for (4.4) in $X$, whose stable and unstable manifolds $\tilde{W}_s^0$ and $\tilde{W}_u^0$ coincide along a homoclinic manifold $\tilde{\Gamma} = \Gamma \times S^1$ in $X$.

For $\alpha \neq 0$, for all $\alpha$ sufficiently near 0, the hyperbolic limit cycle for (4.4) persists as $\tilde{p}_0^0(t, \alpha) = (p_0^0(t, \alpha), \omega t \text{ (mod } 2\pi))$ in $X$, where $p_0^0(t, 0) = \tilde{p}_0^0(t)$. The local stable and unstable manifolds $\tilde{W}_{s,loc}^a$ and $\tilde{W}_{u,loc}^a$ of $\tilde{p}_0^0(t, \alpha)$ also persist, and remain $O(|\alpha|)$-close in $X$ to the local stable and unstable manifolds $\tilde{W}_{s,loc}^0$ and $\tilde{W}_{u,loc}^0$ of $\tilde{p}_0^0(t)$, but globally the homoclinic manifold $\tilde{\Gamma}$ in $X$ may break up in some way (and typically does) for $\alpha \neq 0$.

We define the two-dimensional vector field
\[
f^\perp = \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix},
\]
which is orthogonal to the unperturbed two-dimensional (Hamiltonian) vector field $f$. We define
\[
M(\theta, \alpha) = \beta(\theta, \alpha) \| f^\perp(x^0(0)) \|,
\]
where $\beta(\theta, \alpha)$ is a split function, a signed “horizontal” distance between $\tilde{W}_s^a$ and $\tilde{W}_u^a$.

\[
M_\alpha(\theta, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), F_\alpha(x^0(t), t + \theta/\omega) \rangle dt = \int_{-\infty}^{+\infty} \langle f^\perp(x^0(t)), g(x^0(t), \omega t + \theta) \rangle dt,
\]

\[\tag{4.5}\]
is called a Melnikov integral, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^2$, and $\eta(t) = f^\perp(x^0(t))$.

**Theorem 4.2. (Melnikov’s Method for Nonautonomous Periodic Perturbations)** If $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ are $C^2$ and satisfy conditions (M 1) and
\[
M_\alpha(\theta, 0) = 0, \quad M_{\alpha 0}(\theta, 0) \neq 0 \quad \text{for some } \theta_0 \in S^1,
\]
then for all $\alpha \neq 0$ sufficiently close to 0, the stable and unstable manifolds $\tilde{W}_s^a$ and $\tilde{W}_u^a$ for (4.4) have transverse intersections, and the Poincaré maps $P(\cdot, \alpha)$ for (4.4) have homoclinic points. Furthermore, if $M_\alpha(\theta, 0) \neq 0$ for all $\theta \in S^1$, then for all $\alpha \neq 0$ sufficiently close to 0, there is some open neighbourhood of $\tilde{\Gamma} \cup \{\tilde{p}_0^0(t)\}$ in $X$ where $\tilde{W}_s^a$ and $\tilde{W}_u^a$ do not intersect.

If the perturbed two-dimensional system is autonomous but depends on another parameter $\gamma \in \mathbb{R}^1$,
\[
\dot{x} = F(x, \gamma, \alpha) = f(x) + \alpha g(x, \gamma), \quad x \in \mathbb{R}^2, \quad \gamma \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,
\]
we get a similar result by considering the appropriate Melnikov integral
\[
M_\alpha(\gamma, 0) = \int_{-\infty}^{+\infty} \langle \eta(t), F_\alpha(x^0(t), \gamma, 0) \rangle dt = \int_{-\infty}^{+\infty} \langle f^\perp(x^0(t)), g(x^0(t), \gamma) \rangle dt.
\]

\[\tag{4.7}\]

**Theorem 4.3. (Melnikov’s Method for Autonomous Perturbations)** If $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2$ are $C^2$ and satisfy conditions (M 1) and
\[
M_\alpha(\gamma, 0) = 0, \quad M_{\alpha \gamma}(\gamma, 0) \neq 0 \quad \text{for some } \gamma_0 \in \mathbb{R}^1,
\]
then for all $\alpha \neq 0$ sufficiently close to 0, there is an open neighbourhood of $\Gamma \cup \{p_0^0\}$ in which the family of two-dimensional vector fields (4.6) has homoclinic orbits for a unique $\gamma = \tilde{\gamma}(\alpha) = \gamma_0 + O(|\alpha|)$ and no homoclinic orbit for $\gamma \neq \tilde{\gamma}(\alpha)$. Furthermore, if $M_\alpha(\gamma, 0) \neq 0$ for all $\gamma$, then for all $\alpha \neq 0$ sufficiently small, (4.6) has no homoclinic orbit in some open neighbourhood of $\Gamma \cup \{p_0^0\}$.
The sign of $M_\alpha(\gamma, 0)$ indicates how the stable and unstable manifolds split.

**Discussion of Theorem 4.2.** We want to determine the relative positions of the global stable and unstable manifolds $\tilde{W}_s^\alpha$ and $\tilde{W}_u^\alpha$ of the cycle $\tilde{p}^0(t, \alpha)$ in $X$, in some fixed open neighbourhood of $\tilde{\Gamma} \cup \{\tilde{p}_0^0(t)\}$ in $X$, for $\alpha$ near 0. To make this determination, we use $f^\perp$ given above, to define another, local “vertical” cross-section for the family of three-dimensional vector fields (4.4) for all $\alpha$ near 0,

$$\Sigma^\perp = \{(x, \theta) \in X : x = x^0(0) + \epsilon f^\perp(x^0(0))/\|f^\perp(x^0(0))\|, \theta \in S^1, \epsilon \in \mathbb{R}^1 \text{ near } 0\}.$$

For $\alpha = 0$, $\Sigma^\perp$ has an orthogonal (and therefore transversal) intersection with the homoclinic manifold $\tilde{\Gamma} \subset \tilde{W}_s^0 \cap \tilde{W}_u^0$ in $X$, and it follows that for all $\alpha$ near 0, $\Sigma^\perp$ has transversal intersections with both $\tilde{W}_s^\alpha$ and $\tilde{W}_u^\alpha$ in $X$. 