Definitions: bifurcation value in parameter space, branching diagram (or bifurcation diagram) in parameter-state space, bifurcation diagram (or bifurcation set) in parameter space, local bifurcation.

1. Topological equivalence of families

We compare $m$-parameter families of $n$-dimensional vector fields, and define more precisely what is meant when we say they are “qualitatively the same”. Two families of vector fields $f, g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$,

\[
\frac{dx}{dt} = f(x, \alpha) \quad \text{at} \quad (x_0, \alpha_0) \in \mathbb{R}^n \times \mathbb{R}^m,
\]

and

\[
\frac{dy}{dt} = g(y, \beta) \quad \text{at} \quad (y_0, \beta_0) \in \mathbb{R}^n \times \mathbb{R}^m,
\]

are locally topologically equivalent if there is a local homeomorphism $p : \mathbb{R}^m \to \mathbb{R}^m$ with $p(\alpha_0) = \beta_0$, and an $m$-parameter family of local homeomorphisms $h_{(\alpha)} = h(\cdot, \alpha) : \mathbb{R}^n \to \mathbb{R}^n$, defined for all $\alpha$ near $\alpha_0$, with $h_{(\alpha)}(x_0) = y_0$, that maps the orbits of the first flow for $x$ near $x_0$ at parameter values $\alpha$ near $\alpha_0$ onto the orbits of the second flow for $y = h_{(\alpha)}(x)$ near $y_0$ at parameter values $\beta = p(\alpha)$ near $\beta_0$, preserving the direction of time.

Similarly, we compare $m$-parameter families of $n$-dimensional maps. Two families of maps

\[ x \mapsto f(x, \alpha) \quad \text{at} \quad (x_0, \alpha_0) \in \mathbb{R}^n \times \mathbb{R}^m, \]

and

\[ y \mapsto g(y, \beta) \quad \text{at} \quad (y_0, \beta_0) \in \mathbb{R}^n \times \mathbb{R}^m, \]

are locally conjugate if there is a local homeomorphism $p : \mathbb{R}^m \to \mathbb{R}^m$ with $p(\alpha_0) = \beta_0$, and an $m$-parameter family of local homeomorphisms $h_{(\alpha)} = h(\cdot, \alpha) : \mathbb{R}^n \to \mathbb{R}^n$, defined for all $\alpha$ near $\alpha_0$, such that

\[ h_{(\alpha)}(f(x, \alpha)) = g(h_{(\alpha)}(x), p(\alpha)) \]

for $x$ near $x_0$ at parameter values $\alpha$ near $\alpha_0$.

2. The fold bifurcation for one-dimensional flows

Let $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ be locally defined and sufficiently smooth near $(x_0^0, \alpha^0) \in \mathbb{R}^1 \times \mathbb{R}^1$, and consider the one-parameter family of one-dimensional vector fields

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1. \]

If the four conditions

\[
\begin{align*}
  f(x_0^0, \alpha^0) &= 0, \quad & (F \ 1) \\
  f_x(x_0^0, \alpha^0) &= 0, \quad & (F \ 2) \\
  a &= f_\alpha(x_0^0, \alpha^0) \neq 0, \quad & (F \ 3) \\
  b &= \frac{1}{2} f_{xx}(x_0^0, \alpha^0) \neq 0, \quad & (F \ 4)
\end{align*}
\]

are all satisfied, then the family of vector fields has a fold bifurcation (also known as a saddle-node bifurcation) at $(x_0^0, \alpha^0)$. Expanding the family of vector fields in a Taylor series at $(x_0^0, \alpha^0)$ and using (F 1)–(F 4), we get

\[
\dot{x} = f(x, \alpha) = a(\alpha - \alpha^0) + b(x - x_0^0)^2 + O(|\alpha - \alpha^0|^2 + |\alpha - \alpha^0||x - x_0^0| + |x - x_0^0|^3). 
\]
It can be proved that the higher-order terms $O(|\alpha - \alpha^0|^2 + |\alpha - \alpha^0||x - x_0^0| + |x - x_0^0|^3)$ do not qualitatively affect the local dynamics. In other words, the family of vector fields is locally topologically equivalent to the normal form which is obtained by discarding (or truncating) the specified higher-order terms in the Taylor series:

**Theorem 3.1.** If $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is $C^3$ in an open set containing $(x_0^0, \alpha^0)$ and satisfies conditions (F 1)–(F 4), then

$$\frac{dx}{dt} = f(x, \alpha) \quad \text{at} \quad (x_0^0, \alpha^0)$$

has a fold bifurcation, locally topologically equivalent to the normal form

$$\frac{dy}{d\tau} = a\beta + by^2 \quad \text{at} \quad (0, 0).$$

By rescaling of the variables $\beta$ and $y$, and possibly changing the sign of $\beta$, we may assume $a = 1$ and $b = \pm 1$, and then the normal form can be expressed even more simply as one of two possibilities (the textbook calls this the “topological” normal form for the fold bifurcation)

$$\frac{dy}{d\tau} = \beta \pm y^2.$$

Since there are no higher-order terms (the theorem says they can be transformed away with suitable homeomorphisms), the normal form (either version) is easily studied.

If the family of one-dimensional vector fields has a constraint, so that it has an equilibrium for all parameter values, then it might have a transcritical bifurcation (see Homework Problem Set 3).

3. The symmetric pitchfork bifurcation for one-dimensional flows

Many systems have symmetries, and these can change the type of bifurcations that one expects to see. A simple case of symmetry occurs when each member of a family of one-dimensional vector fields is an odd function of the state variable. Consider the one-parameter family of one-dimensional vector fields

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1.$$

If the symmetry condition

$$f(-x, \alpha) = -f(x, \alpha) \quad \text{for all} \quad x, \alpha \quad \text{(PF 1)}$$

holds (one of two possible types of “$\mathbb{Z}_2$-equivariance”), then

$$f(0, \alpha) = 0$$

(i.e. $x_{[1]}^0(\alpha) = 0$ is an equilibrium) for all $\alpha$. This implies that we can write

$$f(x, \alpha) = x\tilde{f}(x, \alpha),$$

where $\tilde{f}$ is smooth at $(0, \alpha)$. Furthermore, all even-order partial derivatives of $f$ with respect to $x$ must vanish at $x = 0$:

$$f_{xx}(0, \alpha) = 0, \quad f_{xxxx}(0, \alpha) = 0, \quad \ldots.$$

If, in addition, at some specific parameter value $\alpha^0$ we have the conditions

$$f_x(0, \alpha^0) = 0, \quad (PF 2)$$

$$a = f_{x\alpha}(0, \alpha^0) \neq 0, \quad (PF 3)$$

$$b = \frac{1}{a}f_{xx}(0, \alpha^0) \neq 0, \quad (PF 4)$$

then the family of vector fields has a symmetric pitchfork bifurcation at $(0, \alpha^0)$. To study this, we again expand the family of vector fields in a Taylor series, which now has some missing terms due to the symmetry (PF 1). We obtain two more branches of equilibria $x_{[j]}^0(\alpha), \ j = 2, 3$, for $\alpha$ on one side of $\alpha^0$ (depending on the signs of $a$ and $b$), $\alpha$ near $\alpha_0$. 
Theorem 3.2. If \( f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) is \( C^5 \) in an open set containing \((0, \alpha_0)\) and satisfies conditions (PF 1)–(PF 4), then
\[
\frac{dx}{dt} = f(x, \alpha) \quad \text{at } (0, \alpha_0)
\]
has a symmetric pitchfork bifurcation, locally topologically equivalent to the normal form
\[
\frac{dy}{d\tau} = a\beta y + by^3 \quad \text{at } (0, 0).
\]

Again, by rescaling variables, etc. we could assume \( a = 1, b = \pm 1 \) simplifying further to the topological normal form
\[
\frac{dy}{d\tau} = \beta y \pm y^3.
\]

4. The fold bifurcation for one-dimensional maps

If a one-parameter family of one-dimensional maps
\[
x \mapsto g(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,
\]
has a nonhyperbolic fixed point \( x_0^0 \) at some specific parameter value \( \alpha_0 \), then in one state space dimension, there are only two ways this could occur: \( \mu = \pm 1 \), where \( \mu = g_x(x_0^0, \alpha_0) \) is the “eigenvalue” of the \( 1 \times 1 \) linearization. We first consider the case \( \mu = +1 \). If \( g(x, \alpha) \) satisfies
\[
\begin{align*}
g(x_0^0, \alpha_0) &= x_0^0, \quad \text{(FM 1)} \\
g_x(x_0^0, \alpha_0) &= 1, \quad \text{(FM 2)} \\
a &= g_\alpha(x_0^0, \alpha_0) \neq 0, \quad \text{(FM 3)} \\
b &= \frac{1}{2}g_{xx}(x_0^0, \alpha_0) \neq 0, \quad \text{(FM 4)}
\end{align*}
\]
then the family has a **fold** bifurcation for maps. The Taylor series of the family of maps \( x \mapsto g(x, \alpha) \) at \((x_0^0, \alpha_0)\) is
\[
x \mapsto g(x, \alpha) = x + a(\alpha - \alpha^0) + b(x - x_0^0)^2 + O(|\alpha - \alpha^0|^2 + |\alpha - \alpha^0||x - x_0^0| + |x - x_0^0|^3),
\]
and it can be proved that the higher-order terms do not qualitatively affect the local dynamics:

**Theorem 3.3.** If \( g : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) is \( C^3 \) in an open set containing \((x_0^0, \alpha_0)\) and satisfies conditions (FM 1)–(FM 4), then
\[
x \mapsto g(x, \alpha) \quad \text{at } (x_0^0, \alpha_0)
\]
has a fold bifurcation (for maps), locally conjugate to the normal form
\[
y \mapsto y + a\beta + by^2 \quad \text{at } (0, 0).
\]

(Topological normal form \( y \mapsto y + \beta \pm y^2 \).)

Families of maps can also have **transcritical** bifurcations and **symmetric pitchfork** bifurcations. Like the fold bifurcations for maps, the transcritical and symmetric pitchfork bifurcations for maps behave essentially like discrete-time analogues of the corresponding bifurcations for vector fields.
5. The flip (or period doubling) bifurcation for one-dimensional maps

Now we consider the case \( \mu = -1 \). Consider a one-parameter family of one-dimensional maps

\[
x \mapsto g(x, \alpha), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1,
\]

where \( g \) is smooth enough near \((x_0^0, \alpha^0) \in \mathbb{R}^1 \times \mathbb{R}^1\). If it satisfies

\[
g(x_0^0, \alpha^0) = x_0^0, \quad g_x(x_0^0, \alpha^0) = -1, \quad (PD \ 1)
\]

then \( x = x_0^0 \) is a fixed point that is nonhyperbolic when \( \alpha = \alpha^0 \). In this case, the Implicit Function Theorem can be used to solve \( g(x, \alpha) - x = 0 \) to obtain a unique, locally defined, smooth solution \( x = x^0(\alpha) \), a curve \((x^0(\alpha), \alpha)\) of fixed points (i.e. \( g(x^0(\alpha), \alpha) = x^0(\alpha) \)) through \((0, \alpha^0)\). If

\[
-a = \frac{d}{d\alpha} g_x(x^0(\alpha), \alpha) \bigg|_{\alpha = \alpha^0} \neq 0,
\]

then the linearized stability of the fixed point \( x^0(\alpha) \) changes as \( \alpha \) increases through \( \alpha^0 \). Using an equivalent but more conveniently calculated expression for \(-a\), we assume

\[
-a = g_{x\alpha}(x_0^0, \alpha_0) + \frac{1}{2} g_{xx}(x_0^0, \alpha_0) g_{\alpha}(x_0^0, \alpha_0) \neq 0. \quad (PD \ 3)
\]

If we change coordinates by

\[
x = x^0(\alpha) + u
\]

and expand the family in a Taylor series about the fixed point \( x^0(\alpha) \), the family becomes

\[
u \mapsto g_x(x^0(\alpha), \alpha) u + \frac{1}{2} g_{xx}(x^0(\alpha), \alpha) u^2 + \frac{1}{6} g_{xxx}(x^0(\alpha), \alpha) u^3 + O(|u|^4),
\]

where expanding \( \alpha \)-dependent coefficients about \( \alpha^0 \) we have

\[
g_x(x^0(\alpha), \alpha) = -1 - a(\alpha - \alpha^0) + O(|\alpha - \alpha^0|^2),
\]

\[
\frac{1}{2} g_{xx}(x^0(\alpha), \alpha) = b + O(|\alpha - \alpha^0|), \quad b = \frac{1}{2} g_{xx}(x_0^0, \alpha_0)
\]

\[
\frac{1}{6} g_{xxx}(x^0(\alpha), \alpha) = c + O(|\alpha - \alpha^0|), \quad c = \frac{1}{6} g_{xxx}(x_0^0, \alpha_0).
\]

Next, a coordinate change

\[
u = \tilde{h}(y, \alpha) = y + \tilde{\delta}(\alpha) y^2
\]

can be found that “removes” the Taylor series coefficient of the quadratic term in the state variable in the transformed family (see Homework Problem Set 4), to get

\[
y \mapsto \tilde{g}(y, \alpha) = \tilde{g}_y(0, \alpha) y + \frac{1}{6} \tilde{g}_{yyy}(0, \alpha) y^3 + O(|y|^4).
\]

The idea is to choose \( \tilde{\delta}(\alpha) \) carefully, so that in the transformed family we have \( \tilde{g}_{yy}(0, \alpha) = 0 \) for all \( \alpha \) near \( \alpha^0 \). We assume that the cubic coefficient in the transformed map does not vanish,

\[
c + b^2 \neq 0, \quad \text{where} \quad b = \frac{1}{2} g_{xx}(x_0^0, \alpha^0), \quad c = \frac{1}{6} g_{xxx}(x_0^0, \alpha^0). \quad (PD \ 4)
\]

Then the family of maps has a flip bifurcation (also called a period doubling bifurcation) at \((x_0^0, \alpha_0)\).
Theorem 3.4. If \( g : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \) is \( C^4 \) in an open set containing \((x_0^0, \alpha_0)\) and satisfies conditions (PD 1)–(PD 4), then
\[
x \mapsto g(x, \alpha) \quad \text{at } (x_0^0, \alpha_0)
\]
has a flip (or period doubling) bifurcation, locally conjugate to the normal form
\[
y \mapsto -y - a\beta y + (c + b^2)y^3 \quad \text{at } (0, 0).
\]
(The topological normal form \( y \mapsto -y - \beta y \pm y^3 \).)
The second iterate of the map, \( x \mapsto g^2(x, \mu) = g(g(x, \alpha), \alpha) \) is locally conjugate to the second iterate of the normal form
\[
y \mapsto y + 2a\beta y - 2(c + b^2)y^3 + O(|\beta|^2|y| + |\beta||y|^3 + |y|^5),
\]
which has bifurcating nontrivial fixed points, and these correspond to the points belonging to bifurcating nontrivial 2-cycles for the map \( x \mapsto g(x, \alpha) \) itself.

6. Poincaré normal forms

For both vector fields and maps, there is a systematic procedure to “simplify” the lower-order terms of the Taylor series at an equilibrium or a fixed point. Here we summarize the procedure for vector fields.

Let \( H_k \) denote the finite-dimensional vector space of all vector fields \((\mathbb{R}^n \to \mathbb{R}^n)\) whose components are homogeneous polynomials of order \( k \). Assume that 0 is an equilibrium of the vector field \( \dot{x} = f(x), x \in \mathbb{R}^n \), and expand in a Taylor series at the equilibrium,
\[
\dot{x} = Ax + f^{(2)}(x) + f^{(3)}(x) + \cdots,
\]
where \( A = f_x(0) \) is the linearization of the vector field at the equilibrium, and \( f^{(k)} \in H_k \) are the terms of order exactly \( k \), \( k = 2, 3, \ldots \).

We start by “simplifying” the order 2 terms. Introduce a coordinate transformation
\[
x = y + h^{(2)}(y)
\]
where \( h^{(2)} \in H_2 \). For any integer \( k \geq 2 \) we define the linear operator \( L_A : H_k \to H_k \) by
\[
(L_A h^{(k)})(y) = h_y^{(k)}(y)Ay - Ah^{(k)}(y).
\]
For \( k = 2 \), we find the range \( L_A(H_2) \) and a complementary subspace \( \tilde{H}_2 \) (not necessarily unique) in \( H_2 \) so that
\[
H_2 = L_A(H_2) \oplus \tilde{H}_2.
\]
Relative to any fixed complementary subspace, \( f^{(2)} \) has a unique decomposition
\[
f^{(2)} = g^{(2)} + r^{(2)}, \quad g^{(2)} \in L_A(H_2), \quad r^{(2)} \in \tilde{H}_2.
\]
Then since \( g^{(2)} \) belongs to the range of \( L_A \) in \( H_2 \), there exists \( h^{(2)} \in H_2 \) such that the coordinate change removes the \( g^{(2)} \) component of \( f^{(2)} \) and transforms the vector field into
\[
\dot{y} = Ay + r^{(2)}(y) + O(|y|^3),
\]
and in this way the vector field is “simplified” by the coordinate change. The term \( r^{(2)}(y) \) (if nonzero) contains only the resonant terms of order 2, and we say the transformed vector field has been put into Poincaré normal form up to order 2. Using induction we can show that we may transform the vector field into normal form up to any finite order \( m \), \( m \geq 2 \), provided \( f \) is smooth enough.
Theorem 3.5. If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^{m+1} \) in an open set containing 0 and \( f(0) = 0 \), then there exists a coordinate change

\[
x = y + h^{(2)}(y) + \cdots + h^{(m)}(y), \quad h^{(k)} \in H_k,
\]

that transforms the vector field

\[
\dot{x} = Ax + f^{(2)}(x) + \cdots + f^{(m)}(x) + O(\|x\|^{m+1}), \quad f^k \in H_k, \quad k = 2, \ldots, m,
\]

into Poincaré normal form up to order \( m \),

\[
\dot{y} = Ay + r^{(2)}(y) + \cdots + r^{(m)}(y) + O(\|y\|^{m+1}), \quad r^{(k)} \in \tilde{H}_k, \quad k = 2, \ldots, m,
\]

where each \( r^{(k)} \) contains only resonant terms of order \( k \), \( k = 2, \ldots, m \).

A similar procedure can be done for maps, and for families of vector fields or maps.

7. The Hopf bifurcation for two-dimensional flows

A Hopf bifurcation is the bifurcation of limit cycles, associated with a pair of purely imaginary eigenvalues for the linearization of a vector field at an equilibrium in a family of vector fields.

We review a procedure (more background was provided in the lectures) to determine the existence and stability of limit cycles, at a Hopf bifurcation in a one-parameter family of two-dimensional vector fields

\[
\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}^1.
\]  

(3.3)

If the bifurcation conditions

\[
f(x_0^0, \alpha^0) = 0, \quad (H \, 1)
\]

\[
A_0 = f_x(x_0^0, \alpha^0) \text{ has eigenvalues } \lambda_1^0 = i\omega_0, \lambda_2^0 = -i\omega_0, \text{ with } \omega_0 > 0, \quad (H \, 2)
\]

are satisfied, then \( x = x_0^0 \) is an equilibrium that is nonhyperbolic when \( \alpha = \alpha^0 \). In this case, the Implicit Function Theorem can be used to solve \( f(x, \alpha) = 0 \) to obtain unique (locally defined) solutions \( x = x_0^0(\alpha) \), giving a smooth curve \( (x_0^0(\alpha), \alpha) \) of equilibria through \( (x_0^0, \alpha^0) \). Make the simple coordinate shift

\[
x = x_0^0(\alpha) + u
\]  

(I)

to transform the family of vector fields into

\[
\dot{u} = A(\alpha)u + F(u, \alpha),
\]

(3.4)

where

\[
A(\alpha) = f_x(x_0^0(\alpha), \alpha), \quad F(u, \alpha) = f(x_0^0(\alpha) + u, \alpha) - A(\alpha)u = O(\|u\|^2),
\]

so that \( u = 0 \) is now the equilibrium for all \( \alpha \) near \( \alpha^0 \). The matrix \( A(\alpha) \) of the linearization has complex conjugate eigenvalues

\[
\lambda_1(\alpha) = \mu(\alpha) + i\omega(\alpha), \quad \lambda_2(\alpha) = \bar{\lambda}_1(\alpha) = \mu(\alpha) - i\omega(\alpha)
\]

for \( \alpha \) near \( \alpha^0 \), where \( \mu(\alpha) = \text{Re} \lambda_1(\alpha) = \frac{1}{2} \text{tr} A(\alpha) \) and \( \omega(\alpha) = \text{Im} \lambda_1(\alpha) > 0 \), with

\[
\mu(\alpha^0) = 0, \quad \omega(\alpha^0) = \omega_0 > 0.
\]

We assume that when \( \alpha \) increases past \( \alpha^0 \), the eigenvalues cross the imaginary axis with nonzero “speed”,

\[
a = \mu'(\alpha^0) \neq 0,
\]

(H 3)
so that the linearized stability of the equilibrium \( x^0(\alpha) \) changes as \( \alpha \) increases through \( \alpha^0 \).

We now set \( \alpha = \alpha^0 \) in (3.4) to check the final hypothesis (H 4), below. The vector field is now the same as in Example 3.A, done in the lectures. Find a (nonzero) complex eigenvector \( q \) for the linearization,

\[
A_0q = i\omega_0 q, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbb{C}^2,
\]

and also find a complex adjoint eigenvector \( p \),

\[
A_0p = -i\omega_0 p, \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{C}^2,
\]

normalized so that

\[
\langle p, q \rangle = \bar{p}_1 q_1 + \bar{p}_2 q_2 = 1.
\]

Then decompose any \( u \in \mathbb{R}^2 \) uniquely as

\[
u = z_1 q + \bar{z}_1 \bar{q}, \quad z_1 \in \mathbb{C},
\]

substitute into (3.4), take the inner product with the adjoint eigenvector \( p \), and expand the nonlinear part in powers of \( z_1 \) and \( \bar{z}_1 \),

\[
\langle p, f(z_1 q + \bar{z}_1 \bar{q}, \alpha^0) \rangle = \frac{1}{2!} g_{20}(\alpha^0) z_1^2 + \frac{1}{3!} g_{11}(\alpha^0) z_1 \bar{z}_1 + \frac{1}{2^2} g_{02}(\alpha^0) \bar{z}_1^2 + \frac{1}{2!} g_{21}(\alpha^0) z_1^2 \bar{z}_1 + \cdots.
\]

Assume a nonvanishing “cubic normal form coefficient”

\[
b = \text{Re} \frac{c_1(\alpha^0)}{\text{Re} \left[ \frac{1}{2} g_{21}(\alpha^0) + i \frac{1}{2 \omega_0} g_{20}(\alpha^0) g_{11}(\alpha^0) \right] \neq 0}.
\]

Then the original family of vector fields (3.3) has a **Hopf bifurcation** at \( (x_0^0, \alpha^0) \).

**Theorem 3.6.** If \( f : \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^2 \) is \( C^4 \) in an open set containing \( (x_0^0, \alpha^0) \) and satisfies conditions (H 1)–(H 4), then

\[
\frac{dx}{dt} = f(x, \alpha) \quad \text{at } (x_0^0, \alpha^0)
\]

has a Hopf bifurcation, locally topologically equivalent to the normal form

\[
\frac{dy_1}{d\tau} = a\beta y_1 - \omega_0 y_2 + b (y_1^2 + y_2^2) y_1
\]

\[
\frac{dy_2}{d\tau} = \omega_0 y_1 + a\beta y_2 + b (y_1^2 + y_2^2) y_2
\]

at \( (0, 0), 0 \).

(By rescaling of variables and possibly changing the sign of \( \beta \), we could assume \( \omega_0 = 1, a = 1, b = \pm 1 \) and obtain the topological normal form given in the textbook.)

In polar coordinates \( y_1 = \rho \cos \varphi, \ y_2 = \rho \sin \varphi \), the normal form is

\[
\frac{d\rho}{d\tau} = a\beta \rho + b \rho^3,
\]

\[
\frac{d\varphi}{d\tau} = \omega_0,
\]

which is easily studied to determine the existence and stability of limit cycles at the Hopf bifurcation. There exist limit cycles bifurcating from the origin at \( \beta = 0 \), which are asymptotically stable if \( b < 0 \), or unstable if \( b > 0 \).
8. Centre manifold theory

Centre manifold theory is applied to a vector field or map to “reduce” the dynamical system to one of lower dimension. We summarize the theory for a vector field

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n \] (3.7)

(the theory for a map is similar). At a nonhyperbolic equilibrium \( x^0 \), the linearization \( A = f_x(x^0) \) has a centre subspace \( T^c \) of dimension \( n_0 > 0 \) and there is a smooth local centre manifold \( W^c_{loc}(x^0) \).

**Theorem 3.7. (Centre Manifold Theorem)** If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^p \) (\( p > 1 \)) in an open set containing \( x^0 \), \( f(x^0) = 0 \), and \( A = f_x(x^0) \) has \( n_0 > 0 \) eigenvalues \( \lambda_j \) (counting multiplicities) with \( \text{Re} \lambda_j = 0 \), then there exists a \( C^p \) submanifold \( W^c_{loc}(x^0) \) in \( \mathbb{R}^n \), of dimension \( n_0 \), that is locally invariant for (3.7) and tangent to the translated centre subspace \( T^c + x^0 \) at \( x^0 \). Moreover, there is a neighbourhood \( U \) of \( x^0 \) in \( \mathbb{R}^n \), such that if a solution for (3.7) satisfies \( x(t) \in U \) for all \( t \geq 0 \) [for all \( t \leq 0 \)], then \( x(t) \to W^c_{loc}(x^0) \) as \( t \to +\infty \) [as \( t \to -\infty \)].

Suppose the centre, unstable and stable subspaces of the linearization \( A = f_x(x^0) \) have dimensions \( \dim T^c = n_0 > 0 \), \( \dim T^u = n_+ \), \( \dim T^s = n_- \), with \( n_+ = n_+ + n_- > 0 \), \( n_0 + n_\pm = n \). Let \( T^{su} = T^s \oplus T^u \) be the stable-unstable subspace. Then we have

\[ \mathbb{R}^n = T^c \oplus T^{su}, \quad \dim T^c = n_0, \quad \dim T^{su} = n_\pm \]

and there exists a corresponding linear change of coordinates from \( x \in \mathbb{R}^n \) into \( (u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \) so that in the new coordinates \((u, v)\) the equilibrium is the origin \((0, 0)\) and the linearization of the vector field at the equilibrium is block-diagonal,

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
B & O \\
O & C
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
+ 
\begin{pmatrix}
g(u, v) \\
h(u, v)
\end{pmatrix},
(u, v) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm},
\]

(3.8)

where the block \( B \) is an \( n_0 \times n_0 \) real matrix whose eigenvalues all have zero real parts, the block \( C \) is an \( n_\pm \times n_\pm \) real matrix whose eigenvalues all have nonzero real parts, and both nonlinear functions \( g : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_0} \) and \( h : \mathbb{R}^{n_0} \times \mathbb{R}^{n_\pm} \to \mathbb{R}^{n_\pm} \) are locally defined at the origin \((u, v) = (0, 0)\), are \( C^p \) and \( O(||(u, v)||^2) \) (if \( p \geq 2 \)). In these coordinates, the local centre manifold is represented as the graph of a function

\[ v = V(u), \]

where \( V : \mathbb{R}^{n_0} \to \mathbb{R}^{n_\pm} \) is locally defined at \( u = 0 \), is smooth and \( O(||u||^2) \) (i.e. \( V(0) = 0 \) and \( V_u(0) = 0 \)). By the “Reduction Principle” below, the dynamics restricted to \( W^c_{loc}(x^0) \) essentially determine the dynamics of the full system near \( x^0 \).

**Theorem 3.8. (Reduction Principle)** Under the above hypotheses, (3.7) at \( x^0 \), and (3.8) at \( (0, 0) \), are locally topologically equivalent to

\[
\begin{pmatrix}
\dot{u} \\
\dot{v}
\end{pmatrix} =
\begin{pmatrix}
B & O \\
O & C
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
+ 
\begin{pmatrix}
g(u, V(u)) \\
0
\end{pmatrix},
\text{at } (0, 0).
\]

(3.9)

If there is more than one local centre manifold, then all the resulting systems (3.9) with the different \( V \) are locally smoothly equivalent.

In order to find \( V(u) \), we note that the local invariance of the centre manifold implies \( \dot{v}(t) = V_u(u(t))\dot{u}(t) \) for all local solutions \((u(t), v(t))\) of (3.8) with \( v(t) = V(u(t)) \), and therefore

\[ CV(u) + h(u, V(u)) = V_u(u) [Bu + g(u, V(u))] \]
Note that the nonlinear term \( h(u, v) \) in (3.8) is needed in the reduction to (3.9), even though it is absent from (3.9) itself. We may find a series solution for this first-order differential equation for \( V(u) \). Then
\[
\dot{u} = Bu + g(u, V(u)), \quad u \in \mathbb{R}^{n_0}
\]
represents the local dynamics restricted to the centre manifold \( W^c \). It is a good idea to consider this last equation before calculating any Taylor series coefficients of \( V(u) \) explicitly, to see which coefficients are (probably) needed. It would be a mistake not to calculate coefficients that affect the local dynamics, and it would be inefficient to calculate coefficients that do not affect the local dynamics.

A similar procedure can be done for maps, and also for families of vector fields or maps.

**Computation of centre manifolds.** If the dimension of the state space \( n \) is large, it may not be very convenient to make the linear change of variables taking (3.7) into (3.8). An alternate procedure, a projection method, can be more efficient.

For example, suppose the linearization \( A_0 = f_x(x^0) \) has a simple zero eigenvalue and all other eigenvalues \( \lambda_j \) have \( \text{Re} \lambda_j \neq 0 \). Then \( n_0 = 1 \). Let us assume we have already made a coordinate shift (if necessary) in (3.7) so that the nonhyperbolic equilibrium is the origin, \( f(0) = 0 \), and write (3.7) as
\[
\dot{x} = A_0 x + F_0(x), \quad x \in \mathbb{R}^n,
\]
where \( A_0 = f_x(0) \) and \( F_0(x) = f(x) - A_0 x = O(\|x\|^2) \).

Then we find an eigenvector \( q \in \mathbb{R}^n \) for the zero eigenvalue and an adjoint eigenvector \( p \in \mathbb{R}^n \),
\[
Aq = 0, \quad A^T p = 0,
\]
with \( p \) normalized so that
\[
\langle p, q \rangle = p_1 q_1 + \cdots + p_n q_n = 1.
\]
We have the direct sum
\[
\mathbb{R}^n = T^c \oplus T^{su},
\]
and the centre and stable-unstable subspaces have the simple characterizations
\[
T^c = \text{span}\{q\}, \quad T^{su} = \{p\}^\perp,
\]
where we have used the Fredholm Alternative theorem (Appendix B, updated). So we may write any \( x \in \mathbb{R}^n \) uniquely as
\[
x = uq + y, \quad u \in \mathbb{R}^1, \quad y \in T^{su},
\]
where
\[
u = \langle p, x \rangle, \quad y = x - \langle p, x \rangle q.
\]
Define the projection operators \( P^c : \mathbb{R}^n \to \mathbb{R}^n \) and \( P^{su} : \mathbb{R}^n \to \mathbb{R}^n \), by
\[
P^c x = \langle p, x \rangle q, \quad P^{su} = I_n - P^c,
\]
so that \( P^c \) is the projection of \( \mathbb{R}^n \) onto \( T^c \) along \( T^{su} \), and \( P^{su} \) is the projection of \( \mathbb{R}^n \) onto \( T^{su} \) along \( T^c \).

Substituting (**) into (*), apply the projections \( P^c \) and \( P^{su} \) to obtain the equivalent system
\[
\dot{u} = \langle p, F(uq + y) \rangle, \quad (u, y) \in \mathbb{R}^1 \times T^{su},
\]
\[
\dot{y} = Ay + P^{su} F(uq + y),
\]
In this system (†) the local centre manifold \( W^c_{loc}(0) \) takes the form
\[
y = V(u), \quad u \in \mathbb{R}^1; \quad V(0) = 0, \quad V_u(0) = 0,
\]
and by the Reduction Principle, (†) is reduced to the one-dimensional equation
\[
\dot{u} = \langle p, F(uq + V(u)) \rangle, \quad u \in \mathbb{R}^1.
\]
The vector function \( V(u) \) can be found by using a Taylor series expansion for each component, local invariance, and the constraint \( \langle p, V(u) \rangle = 0 \).