1. Existence, uniqueness, smooth dependence

Let $U \subset \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^m$ be an open set, and let $f : U \to \mathbb{R}^n$ be $C^p$ ($p \geq 1$). The basic theorem (Theorem 2.1) on existence, uniqueness, and smooth dependence of solutions states that for any $(t_0, x_0, \alpha) \in U$, the initial value problem

$$\dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0$$

has a unique solution $x(t) = \varphi(t, t_0, x_0, \alpha)$, defined for $t$ belonging to some maximal open interval of existence $J$ that in general depends on $(t_0, x_0, \alpha)$. Furthermore, $\varphi(t, t_0, x_0, \alpha)$ is a $C^p$ function of the “final” time $t \in \mathbb{R}^1$, the initial time $t_0 \in \mathbb{R}^1$, the initial value $x_0 \in \mathbb{R}^n$, and the parameter $\alpha \in \mathbb{R}^m$.

2. Flows (solutions of autonomous systems of differential equations)

If $f$ does not depend explicitly on $t$, the system of ordinary differential equations

$$\dot{x} = f(x)$$

is called autonomous and the solution $x(t) = \varphi(t, 0, x_0)$ to the initial value problem with initial condition $x(0) = x_0$ has some nice properties (e.g. without loss of generality we can take the initial time $t_0 = 0$). In this case, we call $f$ (or the autonomous system $\dot{x} = f(x)$) a vector field, we call $x$-values states, and we call the domain of $f$ state space. We call the collection of all solutions of initial value problems $\varphi(t, 0, x_0)$ the local flow in the state space, generated by the vector field $f$, and we use the notation

$$\varphi^t(x_0) = \varphi(t, 0, x_0),$$

where $\varphi^t$ is the evolution operator. If the vector field is $C^p$ then, for each fixed $t$, the evolution operator $\varphi^t$ is a local $C^p$ diffeomorphism in $\mathbb{R}^n$. If $\varphi^t(x_0)$ is defined for all times $t \in \mathbb{R}$ and all initial states $x_0 \in X$, where $X$ is $\mathbb{R}^n$ or a manifold, then the local flow is called a flow on $X$. Local flows and flows are also called continuous-time dynamical systems, but often we will call them flows, even if they are only local flows.

Other definitions: solution curve, orbit, phase portrait, invariant (also positively invariant, etc.) set, Lyapunov stable, asymptotically stable and unstable invariant set, equilibrium.

3. Maps

A local diffeomorphism in $\mathbb{R}^n$, or a diffeomorphism on $\mathbb{R}^n$

$$x \mapsto g(x),$$

generates a discrete-time dynamical system by the recursion

$$x_{k+1} = g(x_k), \quad k \in \mathbb{Z},$$

consisting of the evolution operator $g^k$, $k \in \mathbb{Z}$ (iterates of $g$ or of its inverse $g^{-1}$). We will call these discrete-time dynamical systems, or the (possibly local) diffeomorphisms themselves, maps.

Other definitions: orbit, phase portrait, invariant (also positively invariant, etc.) set, Lyapunov stable, asymptotically stable and unstable invariant set, fixed point.
4. **Linearization**

**Flows:** If \( x^0 \) is an equilibrium of \( \dot{x} = f(x) \), then the **linearization** (or **variational equation**) at \( x^0 \) is the linear vector field

\[
\dot{u} = Au, \quad \text{where } A = f_x(x^0).
\]

**Maps:** If \( x^0 \) is a fixed point of \( x \mapsto g(x) \), then the **linearization** at \( x^0 \) is the linear map

\[
u \mapsto Bu, \quad \text{where } B = g_x(x^0).
\]

The eigenvalues of \( B \) are called the **multipliers** of \( x^0 \).

**Definitions:** **hyperbolic** equilibria and fixed points.

5. **Topological equivalence and conjugacy**

To compare two different dynamical systems, we would like to be able to decide when they have “qualitatively the same dynamics”. For flows, the appropriate concept is local **topological equivalence**, while for maps, it is local **conjugacy**.

When an equilibrium or fixed point is hyperbolic, linearization gives a reliable qualitative description of the local dynamics. Near a hyperbolic equilibrium or fixed point, one needs to know the locations in the complex plane of the eigenvalues or multipliers of the linearizations, relative to the imaginary axis or unit circle, in order to determine whether two systems are locally topologically equivalent or locally conjugate:

**Theorem 2.2.** **Two local flows at hyperbolic equilibria**, \( x^0 \) for \( \frac{dx}{dt} = f(x) \) and \( y^0 \) for \( \frac{dy}{d\tau} = g(y) \), are locally topologically equivalent if and only if their linearizations at their respective equilibria have the same the dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \) of stable and unstable subspaces, respectively.

**Theorem 2.3.** **Two maps at hyperbolic fixed points**, \( x^0 \) for \( x \mapsto f(x) \) and \( y^0 \) for \( y \mapsto g(y) \), are locally conjugate if and only if their linearizations at their respective fixed points have (i) the same dimensions \( n_- = \dim T^s \) and \( n_+ = \dim T^u \) of stable and unstable subspaces, respectively, and (ii) the same signs of products of all multipliers with \( |\mu| < 1 \) and with \( |\mu| > 1 \).

It is usually easier to compare vector fields rather than (local) flows (in applications a vector field is often given explicitly as a system of differential equations but the flow is essentially all the solutions of the system of differential equations), but there is no easy general way to decide if two flows are locally topologically equivalent by studying only their vector fields (two important exceptions being linear flows, and nonlinear flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice: i) if a vector field is locally **smoothly conjugate** (or locally **smoothly equivalent**) to a second vector field, then their flows are locally topologically equivalent by studying only their vector fields (two important exceptions being linear flows, and nonlinear flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice: i) if a vector field is locally **smoothly conjugate** (or locally **smoothly equivalent**) to a second vector field, then their flows are locally topologically equivalent; ii) if a vector field is locally **orbitally equivalent** to a second vector field, then their flows are locally topologically equivalent; iii) if a vector field is locally **smoothly orbitally equivalent** to a second vector field (i.e. if the vector field is smoothly conjugate to a third vector field that is orbitally equivalent to the second vector field), then their flows are locally topologically equivalent.

6. **Linearization at cycles**

**Flows:** If \( x^0(t) \) is a cycle (or **periodic orbit**) for the flow generated by the vector field \( \dot{x} = f(x) \), the **linearization** (or **variational equation**) of the vector field at \( x^0(t) \) is

\[
\dot{u} = A(t)u, \quad \text{where } A(t) = f_x(x^0(t)).
\]
Since $A(t) = f_x(x^0(t))$ is a continuous periodic $n \times n$ matrix, we can use Floquet theory. If $x^0(t)$ is a cycle for an autonomous system, then one of the Floquet multipliers of the linearization at $x^0(t)$ must be equal to 1, so the Floquet multipliers are 
$$\mu_1, \ldots, \mu_{n-1}, 1$$
(not necessarily all distinct). The $n - 1$ nontrivial Floquet multipliers $\mu_1, \ldots, \mu_{n-1}$ contain linearized stability information. The cycle is hyperbolic if none of these nontrivial Floquet multipliers is on the unit circle $|\mu| = 1$. (A cycle for a flow is a limit cycle if there is an open set that contains the cycle but no other cycles; if a cycle is hyperbolic then it is a limit cycle.)

Maps: $\{x^0_0, \ldots, x^0_{K_0-1}\}$ is a cycle, of period $K_0$, for a map $x \mapsto g(x)$, if and only if each point in the cycle is a fixed point of the $K_0$th iterate $g^{K_0}$. To study the stability of the $K_0$-cycle for $x \mapsto g(x)$, we linearize the map $x \mapsto g^{K_0}(x)$ at any of its $K_0$ fixed points. The cycle is hyperbolic if any of the points $x^0_j$ is a hyperbolic fixed point of $g^{K_0}$.

7. Poincaré maps

Near a cycle $x^0(t)$ for a flow, we can study the fully nonlinear (but perhaps only local) dynamics by means of a Poincaré map, defined in a smooth $(n-1)$-dimensional cross-section $\Sigma$ at a point $x^0_0 = x^0(t_0)$ on the cycle. The Poincaré map is a local diffeomorphism in $\Sigma$, obtained by following the orbit starting from any state sufficiently near $x^0_0$ in $\Sigma$ until the first time (near the period of the cycle) the orbit returns to $\Sigma$. Therefore, $x^0_0$ is a fixed point of the Poincaré map. Using nondegenerate and smooth local coordinates $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ on $\Sigma$, we can express the Poincaré map as $\xi \mapsto P(\xi)$, a map in $\mathbb{R}^{n-1}$, with a fixed point at the coordinate representation $\xi^0_0 \in \mathbb{R}^{n-1}$ of the point $x^0_0 \in \Sigma$. This makes explicit calculations easier.