1. Existence, uniqueness, smooth dependence

Let \( U \subset \mathbb{R}^1 \times \mathbb{R}^n \times \mathbb{R}^m \) be an open set, and let \( f : U \to \mathbb{R}^n \) be \( C^p \) \((p \geq 1)\). The basic theorem (Theorem 2.1) on existence, uniqueness, and smooth dependence of solutions states that for any \((t_0, x_0, \alpha) \in U\), the initial value problem
\[
\dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0
\]
has a unique solution \( x(t) = \varphi(t, t_0, x_0, \alpha) \), defined for \( t \) belonging to some maximal open interval of existence \( J \) that in general depends on \((t_0, x_0, \alpha)\). Furthermore, \( \varphi(t, t_0, x_0, \alpha) \) is a \( C^p \) function of the “final” time \( t \in \mathbb{R}^1 \), the initial time \( t_0 \in \mathbb{R}^1 \), the initial value \( x_0 \in \mathbb{R}^n \), and the parameter \( \alpha \in \mathbb{R}^m \).

2. Flows

If \( f \) does not depend explicitly on \( t \), the system of ordinary differential equations \( \dot{x} = f(x) \) is called autonomous and the solution \( x(t) = \varphi(t, t_0, x_0) \) to the initial value problem
\[
\dot{x} = f(x), \quad x(t_0) = x_0
\]
has some nice properties, e.g. without loss of generality we can always take the initial time to be \( t_0 = 0 \).

In this case, we call \( f \) a vector field, we call \( x \)-values states, and we call the domain of \( f \) state space. We call the collection of the solutions of all initial value problems \( \varphi(t, 0, x_0) \) the local flow in the state space generated by the vector field \( f \), and we use the notation
\[
\varphi^t(x_0) = \varphi(t, 0, x_0),
\]
where \( \varphi^t \) is the evolution operator. If the vector field is \( C^p \) then, for each fixed \( t \), the evolution operator \( \varphi^t \) is a local \( C^p \) diffeomorphism in \( \mathbb{R}^n \). If \( \varphi^t(x_0) \) is defined for all times \( t \in \mathbb{R} \) and all initial states \( x_0 \in X \), where \( X \) is \( \mathbb{R}^n \) or a manifold, then the local flow is called a flow on \( X \). Local flows and flows are also called continuous-time dynamical systems, but often we will call them flows, even if they are only local flows.

Other definitions: solution curve, orbit, phase portrait, invariant (also positively invariant, etc.) set, Lyapunov stable, asymptotically stable and unstable invariant set, equilibrium.

3. Maps

A local diffeomorphism in \( \mathbb{R}^n \), or a diffeomorphism on \( \mathbb{R}^n \)
\[
x \mapsto g(x),
\]
generates a discrete-time dynamical system by the recursion
\[
x_{k+1} = g(x_k), \quad k \in \mathbb{Z},
\]
consisting of the evolution operator \( g^k \), \( k \in \mathbb{Z} \) (iterates of \( g \) or of its inverse \( g^{-1} \)). We will call these discrete-time dynamical systems, or the (possibly local) diffeomorphisms themselves, maps.

Other definitions: orbit, phase portrait, invariant (also positively invariant, etc.) set, Lyapunov stable, asymptotically stable and unstable invariant set, fixed point.
4. Linearization

**Flows:** If $x^0$ is an equilibrium of $\dot{x} = f(x)$, then the linearization (or variational equation) at $x^0$ is the linear vector field

$$\dot{u} = Au,$$ where $A = f_x(x^0)$.

**Maps:** If $x^0$ is a fixed point of $x \mapsto g(x)$, then the linearization at $x^0$ is the linear map

$$u \mapsto Bu,$$ where $B = g_x(x^0)$.

The eigenvalues of $B$ are called the multipliers of $x^0$.

**Definitions:** hyperbolic equilibria and fixed points.

5. Topological equivalence and conjugacy

To compare two different dynamical systems, we would like to be able to decide when they have “qualitatively the same dynamics”. For flows, the appropriate concept is local topological equivalence, while for maps, it is local conjugacy.

When an equilibrium or fixed point is hyperbolic, linearization gives a reliable qualitative description of the local dynamics. Near a hyperbolic equilibrium or fixed point, one needs to know the locations in the complex plane of the eigenvalues or multipliers of the linearizations, relative to the imaginary axis or unit circle, in order to determine whether two systems are locally topologically equivalent or locally conjugate:

**Theorem 2.2.** Two local flows at hyperbolic equilibria, $x^0$ for $\frac{dx}{dt} = f(x)$ and $y^0$ for $\frac{dy}{d\tau} = g(y)$, are locally topologically equivalent if and only if their linearizations at their respective equilibria have the same the dimensions $n_- = \dim T^s$ and $n_+ = \dim T^u$ of stable and unstable subspaces, respectively.

**Theorem 2.3.** Two maps at hyperbolic fixed points, $x^0$ for $x \mapsto f(x)$ and $y^0$ for $y \mapsto g(y)$, are locally conjugate if and only if their linearizations at their respective fixed points have (i) the same dimensions $n_- = \dim T^s$ and $n_+ = \dim T^u$ of stable and unstable subspaces, respectively, and (ii) the same signs of products of all multipliers with $|\mu| < 1$ and with $|\mu| > 1$.

It is usually easier to compare vector fields rather than (local) flows (in applications a vector field is often given explicitly as a system of differential equations but the flow is essentially all the solutions of the system of differential equations), but there is no easy general way to decide if two flows are locally topologically equivalent by studying only their vector fields (two important exceptions being linear flows, and nonlinear flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice: i) if a vector field is locally smoothly conjugate (or locally smoothly equivalent) to a second vector field, then their flows are locally topologically equivalent by studying only their vector fields (two important exceptions being linear flows, and nonlinear flows near hyperbolic equilibria). However, some sufficient conditions can be useful in practice: i) if a vector field is locally smoothly conjugate (or locally smoothly equivalent) to a second vector field, then their flows are locally topologically equivalent; ii) if a vector field is locally orbitally equivalent to a second vector field, then their flows are locally topologically equivalent; iii) if a vector field is locally smoothly orbitally equivalent to a second vector field (i.e. if the vector field is smoothly conjugate to a third vector field that is orbitally equivalent to the second vector field), then their flows are locally topologically equivalent.

6. Linearization at cycles

**For flows:** If $x^0(t)$ is a cycle (or periodic orbit) for the flow generated by the vector field $\dot{x} = f(x)$, the linearization (or variational equation) of the vector field at $x^0(t)$ is

$$\dot{u} = A(t)u,$$ where $A(t) = f_x(x^0(t))$. 

2
Since $A(t) = f_x(x^0(t))$ is a continuous periodic $n \times n$ matrix, we can use Floquet theory. If $x^0(t)$ is a cycle for an autonomous system, then one of the Floquet multipliers of the linearization at $x^0(t)$ must be equal to $1$, so the Floquet multipliers are
\[ \mu_1, \ldots, \mu_{n-1}, 1 \]
(not necessarily all distinct). The $n - 1$ nontrivial Floquet multipliers $\mu_1, \ldots, \mu_{n-1}$ contain linearized stability information. The cycle is hyperbolic if none of these nontrivial Floquet multipliers is on the unit circle $|\mu| = 1$. (A cycle for a flow is a limit cycle if there is an open set that contains the cycle but no other cycles; if a cycle is hyperbolic then it is a limit cycle.)

For maps: \{x^0_0, \ldots, x^0_{K_0-1}\} is a cycle, of period $K_0$, for a map $x \mapsto g(x)$, if and only if each point in the cycle is a fixed point of the $K_0$th iterate $g^{K_0}$. To study the stability of the $K_0$-cycle for $x \mapsto g(x)$, we linearize the map $x \mapsto g^{K_0}(x)$ at any of its $K_0$ fixed points. The cycle is hyperbolic if any of the points $x^0_j$ is a hyperbolic fixed point of $g^{K_0}$.

7. Poincaré maps

Near a cycle $x^0(t)$ for a flow, we can study the fully nonlinear (but perhaps only local) dynamics by means of a Poincaré map, defined in a smooth $(n-1)$-dimensional cross-section $\Sigma$ at a point $x^0_0 = x^0(t_0)$ on the cycle. The Poincaré map is a local diffeomorphism in $\Sigma$, obtained by following the orbit starting from any state sufficiently near $x^0_0$ in $\Sigma$ until the first time (near the period of the cycle) the orbit returns to $\Sigma$. Therefore, $x^0_0$ is a fixed point of the Poincaré map. Using nondegenerate, smooth local coordinates $\xi = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ on $\Sigma$, we can express the Poincaré map as $\xi \mapsto P(\xi)$, a map in $\mathbb{R}^{n-1}$, with a fixed point at the coordinate representation $\xi^0_0 \in \mathbb{R}^{n-1}$ of the point $x^0_0 \in \Sigma$. This makes explicit calculations easier. The stability of the cycle $x^0(t)$ for the flow corresponds to the stability of the fixed point $\xi^0_0$ for the Poincaré map $P$. Linearized stability of the cycle is determined by the multipliers (eigenvalues) $\mu_1, \ldots, \mu_{n-1}$ of the linearization of the Poincaré map at $\xi^0_0$, the $(n-1) \times (n-1)$ matrix $P_\xi(\xi^0_0)$. These multipliers are independent of the choice of the point $x^0_0$ on the cycle, of the choice of the cross-section $\Sigma$ at $x^0_0$, and of the choice of the coordinates $\xi$ on $\Sigma$.

Theorem 2.4. The nontrivial Floquet multipliers of the linearization $\dot{u} = f_x(x^0(t))u$, of the vector field $\dot{x} = f(x)$ at its cycle $x^0(t)$ in $\mathbb{R}^n$ are the same as the multipliers of the linearization $v \mapsto P_\xi(\xi^0_0)v$, of the Poincaré map $\xi \mapsto P(\xi)$ at its corresponding fixed point $\xi^0_0$ in $\mathbb{R}^{n-1}$.

Thus, a cycle for a flow is hyperbolic if and only if the corresponding fixed point for a Poincaré map is hyperbolic.

8. Local stable and unstable manifolds

If $x^0$ is an equilibrium of a flow or a fixed point of a map, then its stable set $W^s(x^0)$ is the set of all states whose orbits approach $x^0$ in forward time, while its unstable set $W^u(x^0)$ is the set of all states whose orbits approach $x^0$ in backward time. Both these sets are invariant. If $x^0$ is hyperbolic, then more can be said:

Theorem 2.5. If $f$ is $C^p$ ($p \geq 1$) and $x^0$ is a hyperbolic equilibrium for $\dot{x} = f(x)$, then the intersections of $W^s(x^0)$ and $W^u(x^0)$ with a sufficiently small open neighbourhood of $x^0$ contain $C^p$ submanifolds $W^s_{\text{loc}}(x^0)$ and $W^u_{\text{loc}}(x^0)$ of dimensions $n_- = \dim T^s$ and $n_+ = \dim T^u$, respectively. The submanifolds $W^s_{\text{loc}}(x^0)$ and $W^u_{\text{loc}}(x^0)$ are tangent at $x^0$ to $T^s + x^0$ and $T^u + x^0$, where $T^s$ and $T^u$ are the stable and unstable subspaces of the linearized vector field at $x^0$, respectively.

Theorem 2.6. If $g$ is $C^p$ ($p \geq 1$) and $x^0$ is a hyperbolic fixed point for $x \mapsto g(x)$, then the intersections of $W^s(x^0)$ and $W^u(x^0)$ with a sufficiently small open neighbourhood of $x^0$ contain $C^p$ submanifolds $W^s_{\text{loc}}(x^0)$ and $W^u_{\text{loc}}(x^0)$ of dimensions $n_- = \dim T^s$ and $n_+ = \dim T^u$, respectively. The submanifolds $W^s_{\text{loc}}(x^0)$ and $W^u_{\text{loc}}(x^0)$ are tangent at $x^0$ to $T^s + x^0$ and $T^u + x^0$, where $T^s$ and $T^u$ are the stable and unstable subspaces of the linearized map at $x^0$, respectively.
If the equilibrium or fixed point $x^0$ is hyperbolic, then letting all states in the positively invariant local stable manifold $W^{s}_{loc}(x^0)$ evolve backwards in time we recover $W^s(x^0)$, and similarly letting all states in the negatively invariant local unstable manifold $W^{u}_{loc}(x^0)$ evolve forwards in time we recover $W^u(x^0)$. This implies that $W^s(x^0)$ and $W^u(x^0)$ are locally $C^p$ submanifolds. For this reason (if $x^0$ is hyperbolic) the stable and unstable sets $W^s(x^0)$ and $W^u(x^0)$ are often referred to as the (global) stable and unstable manifolds of $x^0$, respectively. They are important features of the dynamics. Sometimes these sets globally have complicated features.

9. Hamiltonian systems

A **Hamiltonian system** is a vector field in the even-dimensional state space $\mathbb{R}^{2n}$, generated by a real-valued $C^{p+1}$ ($p \geq 1$) **Hamiltonian function** $H$ with domain in $\mathbb{R}^{2n}$, of the form

$$\dot{x} = H_y(x, y), \quad \dot{y} = -H_x(x, y),$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. In a Hamiltonian system, all solutions $(x(t), y(t))$ remain on level sets $H(x, y) = \text{constant}$. This fact (“conservation of energy”) makes determining the global phase portrait especially easy in the case of $n = 1$.

10. Lyapunov functions

A **Lyapunov function** $L(x)$ for a vector field $\dot{x} = f(x)$ can be useful, in those cases where one can be found. Not only can a Lyapunov function be used to study the stability of a equilibrium, but it can also be useful in constructing a **trapping region** for a flow, and it can help in determining the global dynamics of a flow.