1. Linear homogeneous systems of differential equations

See Appendix A. Review of Differential Calculus. Let \( J \subset \mathbb{R} \) be an open interval and let \( A(t) \) be a continuous real \( n \times n \) matrix of coefficient functions for \( t \in J \). Then the linear homogeneous system of ODEs
\[
\dot{x} = A(t)x, \quad x \in \mathbb{R}^n
\]
always has \( n \) linearly independent real solutions on \( J \) (for more background, see almost any undergraduate ODE textbook). Arranging these solutions as the columns of a real \( n \times n \) matrix, we get a fundamental matrix \( \Psi(t) \). A fundamental matrix satisfies
\[
\dot{\Psi} = A(t)\Psi, \quad \det(\Psi(t)) \neq 0 \text{ for all } t \in J.
\]
In terms of a fundamental matrix, the unique solution of the initial value problem
\[
\dot{x} = A(t)x, \quad x(t_0) = x_0,
\]
where \( t_0 \in J \), can be written as
\[
x(t) = \Psi(t)\Psi(t_0)^{-1}x_0.
\]
Thus, it is always possible to find the fundamental matrix \( \Phi(t) = \Phi(t; t_0) \) satisfying \( \Phi(t_0; t_0) = I_n \), where \( I_n \) denotes the \( n \times n \) identity matrix.

2. Linear vector fields (autonomous linear homogeneous systems of differential equations)

If \( A \) is a constant real \( n \times n \) matrix, then the linear homogeneous system (with constant coefficients)
\[
\dot{x} = Ax, \quad x \in \mathbb{R}^n
\]
is called autonomous (the system is also called a linear vector field in this case) and the fundamental matrix \( \Phi(t; 0) \) for the system that satisfies \( \Phi(0; 0) = I_n \) is the exponential matrix (or linear flow or linear evolution operator)
\[
\Phi(t; 0) = e^{At}.
\]
See Appendix B. Some Linear Algebra. It is possible to find the linear flow \( e^{At} \) explicitly, by finding a basis of generalized eigenvectors of \( A \) and determining the linear nonsingular coordinate changes that take the matrix \( A \) into its Jordan normal form (which is a nonreal complex matrix if \( A \) has any nonreal complex eigenvalues) or its real normal form (which is a real matrix).

3. Stable, centre and unstable subspaces, hyperbolicity, asymptotic behaviour

Definitions: (dynamically) invariant set, positively invariant, negatively invariant, locally invariant.

If \( A \) is a constant real \( n \times n \) matrix, then qualitative properties of the linear flow \( e^{At} \), generated by the linear vector field
\[
\dot{x} = Ax
\]
are determined by the real parts \( \text{Re } \lambda_j \) of the eigenvalues \( \lambda_j \) of the matrix of coefficients \( A \). The stable subspace \( T^s \), the centre subspace \( T^c \), the unstable subspace \( T^u \), and the direct sum decomposition
\[
\mathbb{R}^n = T^s \oplus T^c \oplus T^u
\]
for $A$ give much of the qualitative information we are usually interested in, such as stability. All three subspaces $T^s$, $T^c$, $T^u$ are invariant.

An invariant set $\Lambda$ is (a) Liapunov stable if for any open set $U$ containing $\Lambda$, there exists an open set $V$ containing $\Lambda$ such that $x(0) \in V$ implies $x(t) \in U$ for all $t \in [0, +\infty)$; (b) asymptotically stable if it is Lyapunov stable and there exists an open set $W$ containing $\Lambda$ such that $x(0) \in W$ implies $x(t) \to \Lambda$ as $t \to +\infty$; (c) unstable if it is not Lyapunov stable.

The set consisting of an equilibrium is invariant. If $T^c = \{0\}$ then the linear vector field is hyperbolic (and, since zero is not an eigenvalue, the equilibrium $x_0 = 0$ is unique). If $T^c = \{0\}$ and $T^u = \{0\}$ then the equilibrium $0$ is asymptotically stable. If $T^u \neq \{0\}$ then the equilibrium $0$ is unstable.

4. Linear maps (autonomous linear homogeneous systems of difference equations)

A constant real $n \times n$ matrix $B$ gives a linear map

$$x \mapsto Bx, \quad x \in \mathbb{R}^n$$

(often, $Bx$, or $B$, is called the linear map if the context is understood) which is a linear diffeomorphism if $\det B \neq 0$. The integer powers $B^k$ of $B$, $k \in \mathbb{Z}$, give a (discrete-time) linear evolution operator. The map, or the corresponding evolution operator, can be considered as an example of a discrete-time dynamical system. Qualitative properties of this dynamical system are determined by the moduli (absolute values) $|\mu_j|$ of the multipliers (eigenvalues) $\mu_j$ of $B$. The stable subspace $T^s$, the centre subspace $T^c$, the unstable subspace $T^u$, and the direct sum decomposition

$$\mathbb{R}^n = T^s \oplus T^c \oplus T^u$$

for $B$ give most of the information we are usually interested in (but the definitions of the stable, centre and unstable subspaces are different than for linear vector fields). If $T^c = \{0\}$ then the linear diffeomorphism is hyperbolic. If $T^c = \{0\}$ and $T^u = \{0\}$ then the fixed point $x = 0$ is asymptotically stable. If $T^u \neq \{0\}$ then the fixed point $0$ is unstable.

5. Floquet theory (periodic nonautonomous linear homogeneous systems of differential equations)

If $A(t)$ is a continuous periodic real $n \times n$ matrix with period $T_0 > 0$,

$$A(t + T_0) = A(t) \quad \text{for all } t \in \mathbb{R},$$

then to determine the qualitative properties of the linear homogeneous system

$$\dot{x} = A(t)x$$

one chooses some $t_0 \in \mathbb{R}$, finds the fundamental matrix $\Phi(t, t_0)$ that satisfies $\Phi(t_0; t_0) = I_n$, and forms the monodromy matrix at $t_0$

$$M = \Phi(t_0 + T_0; t_0).$$

The eigenvalues $\mu_j$ of the monodromy matrix $M$ do not depend on which initial time $t_0$ is used, and these eigenvalues are called the Floquet multipliers of the system. These determine the qualitative properties of the system. A Floquet exponent is a number $\lambda_j$ such that $e^{\lambda_j} = \mu_j$.

If $A(t)$ is not constant, it can be difficult to find the fundamental matrix $\Phi(t, t_0)$ explicitly. However if $A(t)$ depends continuously on a parameter, then so do the Floquet multipliers. This fact can sometimes be used to determine qualitative properties of the system, by perturbation arguments.