1. Consider the vector field in the plane

\[ \begin{align*}
\dot{x}_1 &= 2x_1^3 - 2x_1^2x_2 + x_1^4, \\
\dot{x}_2 &= 2x_1 - 2x_2 + 2x_2^2.
\end{align*} \]

Observe that the origin \((x_1, x_2) = (0, 0)\) is an equilibrium. Use linear stability analysis to study the stability of the origin. Then use centre manifold theory to determine the dynamics near the origin, including determining the stability of the origin. Draw a two-dimensional phase portraits.

2. Consider the two-parameter family of vector fields in the plane

\[ \begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= \beta_1 + \beta_2 y_1 + y_1^2 + y_1 y_2,
\end{align*} \]

where \(\beta_1, \beta_2\) are real parameters. (When \(\beta_1 = 0\) and \(\beta_2 = 0\), this is the truncated normal form from Homework Problem Set 4, Question 3, with \(a = 1\) and \(b = 1\).)

(a) Find all equilibria, and determine their linearized stabilities (classified as hyperbolic sink, hyperbolic source, hyperbolic saddle, or nonhyperbolic of various subtypes) depending on \(\beta_1, \beta_2\). Summarize your results so far, with a diagram in the \((\beta_1, \beta_2)\) parameter plane, showing the number and classification of equilibria in different regions of the parameter plane.

(b) Verify that the two curves, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, FB\(_1\): \(\beta_1 = (\beta_2)^2/4, \beta_2 > 0\), and FB\(_2\): \(\beta_1 = (\beta_2)^2/4, \beta_2 < 0\), correspond to the existence of an equilibrium whose linearization has a simple zero eigenvalue. Then for fixed \(\beta_2 > 0\) or \(\beta_2 < 0\), and considering \(\beta_1\) as a bifurcation parameter, use centre manifold theory and Theorem 3.1 to show that fold bifurcations occur on both curves FB\(_1\) and FB\(_2\), as the parameter \(\beta_1\) increases through the bifurcation value \((\beta_2)^2/4\). Determine the local dynamics up to local topological equivalence. Draw six representative two-dimensional local phase portraits in some convenient coordinates, for parameter values \((\beta_1, \beta_2)\) near the curves FB\(_1\) and FB\(_2\) \((\beta_1 < (\beta_2)^2/4, \beta_1 = (\beta_2)^2/4, \beta_1 > (\beta_2)^2/4; \text{with } \beta_2 > 0, \text{ or with } \beta_2 < 0)\). Also attempt to draw two-dimensional global phase portraits in the original \((y_1, y_2)\) coordinates (at this stage the global phase portraits will be rather incomplete).

(c) Verify that the curve, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, HB: \(\beta_1 = 0, \beta_2 < 0\), corresponds to the existence of nonhyperbolic linear centres. Fixing \(\beta_2 < 0\), and considering \(\beta_1\) as a bifurcation parameter, use Theorem 3.6 to show that Hopf bifurcations occur on the curve HB, as the parameter \(\beta_1\) increases through the bifurcation value 0. Determine the local dynamics up to local topological equivalence, including existence and stability of limit cycles for parameter values on one side or the other of the curve HB (find which side). Draw three representative two-dimensional local phase portraits in some convenient coordinates, for parameter values \((\beta_1, \beta_2)\) near the curve HB \((\beta_1 < 0, \beta_1 = 0, \beta_1 > 0, \text{ with } \beta_2 < 0)\). Also attempt to draw global phase portraits in the original \((y_1, y_2)\) coordinates, filling in some more details to the first attempts made in part (b).

(d) It can be proved (see Question 3, below) that there is a curve, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, SLB: \(\beta_1 = \beta_1^{SLB}(\beta_2)\), in the region \(\beta_1 < (\beta_2)^2/4\), that corresponds to homoclinic (or saddle loop) bifurcations as \(\beta_1\) increases through \(\beta_1^{SLB}(\beta_2)\) for fixed
\(\beta_2\). It can also be proved that there is never more than one cycle, and that there are no bifurcations other than those on the four curves \(\text{FB}_1,\text{FB}_2,\text{HB},\text{SLB}\). Given these facts, decide where the curve \(\text{SLB}\) can fit in the \((\beta_1,\beta_2)\)-plane relative to the curves \(\text{FB}_1,\text{FB}_2,\text{HB}\), and draw the four curves \(\text{FB}_1,\text{FB}_2,\text{HB},\text{SLB}\) on the same diagram in the \((\beta_1,\beta_2)\)-plane. Draw three (conjectured) two-dimensional phase portraits for parameter values \((\beta_1,\beta_2)\) near the curve \(\text{SLB}\) \((\beta_1 < \beta_{1\text{SLB}}(\beta_2), \beta_1 = \beta_{1\text{SLB}}(\beta_2), \beta_1 > \beta_{1\text{SLB}}(\beta_2) )\), giving as much justification as possible for your conjectures, including checking that the parameter values near your conjectured curve \(\text{SLB}\) and the corresponding phase portraits are consistent with Theorem 4.1.

3. In Question 2, assume \(\beta_1 < (\beta_2)^2/4\), let \((y_1^0, y_2^0) (y_2^0 = 0)\) denote the “left” equilibrium whose linearization has a positive determinant, and make the coordinate shift
\[
y_1 = y_1^0 + u_1, \quad y_2 = y_2^0 + u_2;
\]
to obtain
\[
u_1 = u_2, \quad \dot{u}_2 = -\alpha^2 u_1 + y_1^0 u_2 + u_1^2 + u_1 u_2,
\]
where \(\alpha > 0\) is defined by
\[
\alpha^2 = \sqrt{(\beta_2)^2 - 4\beta_1}.
\]
Then make the rescalings
\[
u_1 = \alpha^2 v_1, \quad \nu_2 = \alpha^3 v_2, \quad y_1^0 = \alpha^2 \gamma, \quad t = \tau/\alpha,
\]
where \(\gamma\) is a constant, to get
\[
\frac{dv_1}{d\tau} = v_2 \quad \frac{dv_2}{d\tau} = -v_1 + v_1^2 + \alpha (\gamma v_2 + v_1 v_2).
\]
(*)
In the limit \(\alpha \to 0\) this system becomes Hamiltonian (but because of the time rescaling, this limit is singular with respect to the original system). For the system (*) with \(\alpha = 0\), find a suitable Hamiltonian function and sketch the global phase portrait. Verify that when \(\alpha = 0\),
\[
(v_1^0(\tau), v_2^0(\tau)) = \left(1 - \frac{3}{2} \, \text{sech}^2 \left(\frac{\tau}{2}\right), \frac{3}{2} \, \text{sech}^2 \left(\frac{\tau}{2}\right) \, \tanh \left(\frac{\tau}{2}\right)\right)
\]
is a homoclinic orbit for (*), and use Melnikov’s method to show that when \(\alpha > 0\) and sufficiently small, there exists a homoclinic orbit for (*) when \(\gamma = \gamma_{\text{SLB}}(\alpha)\), where
\[
\gamma_{\text{SLB}}(\alpha) = \gamma_0 + O(|\alpha|),
\]
and the homoclinic orbit breaks when \(\gamma \neq \gamma_{\text{SLB}}(\alpha)\). Find an explicit numerical value for the leading-order constant \(\gamma_0\). Finally, transform back into the original parameters \(\beta_1\) and \(\beta_2\) to obtain the curve \(\text{SLB}\) of Question 2(d) in the form
\[
\beta_1 = \beta_{1\text{SLB}}(\beta_2) = C \beta_2^2 + O(|\beta_2|^q).
\]
Find explicit values for the constant \(C \neq 0\) and power \(q > 2\).