1. Consider the vector field in the plane

\[ \dot{x}_1 = 2x_1^3 - 2x_1^2x_2 + x_1^4, \quad \dot{x}_2 = 2x_1 - 2x_2 + 2x_2^2. \]

Observe that the origin \((0, 0)\) is an equilibrium. What does linearized stability analysis imply about the stability of this equilibrium? Use centre manifold theory (Theorems 3.7 and 3.8) to determine the dynamics near the equilibrium, including determining the stability of the equilibrium. Draw a two-dimensional phase portrait in some convenient coordinates, and also in the original \((x_1, x_2)\) coordinates.

2. Consider the two-parameter family of vector fields in the plane

\[ \dot{y}_1 = y_2, \quad \dot{y}_2 = \beta_1 + \beta_2y_1 + y_1^2 + y_1y_2, \]

where \(\beta_1, \beta_2\) are real parameters. (When \(\beta_1 = 0\) and \(\beta_2 = 0\), this is the truncated Poincaré normal form from Homework Problem Set 4, Question 3, with \(a = 1\) and \(b = 1\).)

(a) Find all equilibria, and for each equilibrium determine its linearized stability (classified as hyperbolic sink, hyperbolic source, hyperbolic saddle, or nonhyperbolic) depending on \(\beta_1, \beta_2\). Summarize your results so far, with a diagram in the \((\beta_1, \beta_2)\) parameter plane, showing the number and classification of equilibria in different regions of the parameter plane.

(b) Verify that the two curves, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, \(F_1: \beta_1 = (\beta_2)^2/4, \beta_2 > 0, \) and \(F_2: \beta_1 = (\beta_2)^2/4, \beta_2 < 0, \) correspond to the existence of an equilibrium whose linearization has a simple zero eigenvalue. Then for fixed \(\beta_2 > 0\) or \(\beta_2 < 0\), and considering \(\alpha = \beta_1\) as a bifurcation parameter, use centre manifold theory applied the extended 3-dimensional system for \((\alpha, y_1, y_2)\) \(\in \mathbb{R}^3\), and Theorem 3.1 to show that fold bifurcations occur on both curves \(F_1\) and \(F_2\), as the parameter \(\alpha = \beta_1\) increases through the bifurcation value \((\beta_2)^2/4\). Determine the local dynamics, and draw six representative 2-dimensional local phase portraits in some convenient coordinates, correct up to local topological equivalence, for parameter values \((\beta_1, \beta_2)\) near the curves \(F_1\) and \(F_2\) \((\beta_1 < (\beta_2)^2/4, \beta_1 = (\beta_2)^2/4, \beta_1 > (\beta_2)^2/4; \) with \(\beta_2 > 0\), or with \(\beta_2 < 0\). Also, for \(\beta_2 < 0\) only, draw three 2-dimensional phase portraits in the original \((y_1, y_2)\) coordinates for \((\beta_1, \beta_2)\) near the curve \(F_2\) (you can use numerical simulations, but are not required to).

(c) Verify that the curve, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, \(H: \beta_1 = 0, \beta_2 < 0, \) corresponds to the existence of nonhyperbolic linear centres. Fix \(\beta_2 < 0\), and considering \(\alpha = \beta_1\) as a bifurcation parameter, use Theorem 3.6 to show that Hopf bifurcations occur on the curve \(H\), as the parameter \(\alpha = \beta_1\) increases through the bifurcation value 0. Determine the local dynamics up to local topological equivalence, including existence and stability of limit cycles for parameter values on one side or the other of the curve \(H\) (find which side). Draw three representative 2-dimensional local phase portraits in some convenient coordinates near the critical equilibrium, correct up to local topological equivalence, for parameter values \((\beta_1, \beta_2)\) near the curve \(H\) \((\beta_1 < 0, \beta_1 = 0, \beta_1 > 0; \) with \(\beta_2 < 0\)). Also draw global phase portraits in the original \((y_1, y_2)\) coordinates for \((\beta_1, \beta_2)\) near \(H\), showing the local flow and both equilibria.
(d) It can be proved (see Question 3, below) that there is a curve, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, SL: \(\beta_1 = \beta_1^{SL}(\beta_2)\), in the open region \(\beta_1 < (\beta_2)^2/4\), that corresponds to homoclinic (or saddle loop) bifurcations as \(\beta_1\) increases through \(\beta_1^{SL}(\beta_2)\) for fixed \(\beta_2\). It can also be proved that there is never more than one cycle, and that there are no bifurcations other than those on the four curves \(F_1, F_2, H, SL\). Given these facts, determine all the constraints on where the curve SL can fit in the \((\beta_1, \beta_2)\)-plane relative to the other curves \(F_1, F_2\) and \(H\), and draw the four curves \(F_1, F_2, H\), and (conjectured) SL on the same diagram in the \((\beta_1, \beta_2)\)-plane. Draw three (conjectured) 2-dimensional phase portraits in the \((y_1, y_2)\) coordinates for parameter values \((\beta_1, \beta_2)\) near the curve SL \((\beta_1 < \beta_1^{SL}(\beta_2), \beta_1 = \beta_1^{SL}(\beta_2), \beta_1 > \beta_1^{SL}(\beta_2))\), giving as much justification as possible for your conjectures, including checking that the parameter values near your conjectured curve SL and the corresponding phase portraits are consistent with Theorem 4.1.

3. In Question 2, assume \(\beta_1 < (\beta_2)^2/4\), let \(y^0(\beta) = (y^0_1, 0) \in \mathbb{R}^2\) denote the “left” equilibrium (whose linearization has a positive determinant), and make the coordinate shift

\[
y = y^0(\beta) + u
\]

in \(\mathbb{R}^2\) to obtain

\[
\frac{du_1}{dt} = u_2, \quad \frac{du_2}{dt} = -\alpha^2 u_1 + y^0_1 u_2 + u_1^2 + u_1 u_2,
\]

where \(\alpha > 0\) is defined by

\[
\alpha^2 = \sqrt{(\beta_2)^2 - 4\beta_1}.
\]

Then make the rescalings

\[
u_1 = \alpha^2 v_1, \quad u_2 = \alpha^3 v_2, \quad y^0_1 = \alpha^2 \gamma, \quad t = \frac{s}{\alpha},
\]

where \(\gamma\) is a constant, to get

\[
\frac{dv_1}{ds} = v_2, \quad \frac{dv_2}{ds} = -v_1 + v_2 + \alpha(\gamma v_2 + v_1 v_2).
\]

In the limit \(\alpha \to 0\) this system becomes Hamiltonian (but because of the time rescaling, this limit is singular with respect to the original system). Setting \(\alpha = 0\) in the system (*), find a suitable Hamiltonian function and sketch the global phase portrait.

Verify that for \(\alpha = 0\),

\[
(v_1^0(s), v_2^0(s)) = \left(1 - \frac{3}{2} \text{sech}^2 \left(\frac{s}{2}\right), \frac{3}{2} \text{sech}^2 \left(\frac{s}{2}\right) \tanh \left(\frac{s}{2}\right)\right)
\]

is a homoclinic orbit for (*), and use Melnikov’s method to show that for all sufficiently small \(\alpha > 0\), there exists a homoclinic orbit for (*) if \(\gamma = \gamma^{SL}(\alpha)\), a unique value given by the Implicit Function Theorem, where

\[
\gamma^{SL}(\alpha) = \gamma_0 + O(|\alpha|),
\]

and the homoclinic orbit breaks when \(\gamma \neq \gamma^{SL}(\alpha)\). Find an explicit numerical value for the leading-order constant \(\gamma_0\). Finally, transform \(\gamma = \gamma^{SL}(\alpha)\) back into the original parameters \(\beta_1\) and \(\beta_2\), to obtain the curve SL of Question 2(d) in the form

\[
\beta_1 = \beta_1^{SL}(\beta_2) = C(\beta_2)^2 + O(|\beta_2|^q).
\]

Find explicit values for the leading-order coefficient \(C \neq 0\) and higher-order power \(q > 2\).