1. Consider the vector field in the plane

\[ \dot{x}_1 = 2x_1^3 - 2x_1^2x_2 + x_1^4, \quad \dot{x}_2 = 2x_1 - 2x_2 + 2x_1^2. \]

Observe that the origin \((0, 0)\) is an equilibrium. What does linearized stability analysis imply about the stability of this equilibrium? Use centre manifold theory (Theorems 3.7 and 3.8, etc.) to determine the dynamics near the equilibrium, including determining the stability about the stability of this equilibrium? Draw a two-dimensional phase portrait in some convenient coordinates, correct up to local topological equivalence, and also a phase portrait in the original \((x_1, x_2)\) coordinates.

2. Consider the two-parameter family of vector fields in the plane

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = \beta_1 + \beta_2 x_1 + x_1^2 - x_1 x_2, \tag{2.1} \]

where \(\beta_1, \beta_2\) are real parameters.

(a) Find all equilibria for (2.1), and for each equilibrium determine its linearized stability (classify as hyperbolic sink, hyperbolic source, hyperbolic saddle, or nonhyperbolic) depending on \(\beta_1, \beta_2\). Summarize your results so far, with a diagram in the \((\beta_1, \beta_2)\) parameter plane, showing the number and classification of equilibria in different regions of the parameter plane.

(b) Verify from part (a) that two curves, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, \(F_1: \beta_1 = \frac{1}{4}(\beta_2)^2, \beta_2 > 0\), and \(F_2: \beta_1 = \frac{1}{4}(\beta_2)^2, \beta_2 < 0\), correspond to the existence of an equilibrium for (2.1) whose linearization has a simple zero eigenvalue. Then for fixed \(\beta_2 > 0\) or \(\beta_2 < 0\), and considering \(\beta_1 = \frac{1}{4}(\beta_2)^2 + \alpha, \alpha \) near 0, as a bifurcation parameter, use centre manifold theory applied the extended 3-dimensional system for \((\alpha, y_1, y_2) \in \mathbb{R}^3\), where \(x_1 = -\frac{1}{2}\beta_2 + y_2\) and \(x_2 = y_2\), together with Theorem 3.1 to show that fold bifurcations occur on both curves \(F_1\) and \(F_2\), as the parameter \(\beta_1\) increases through the bifurcation value \(\frac{1}{4}(\beta_2)^2\). Determine the local dynamics, and draw six representative two-dimensional local phase portraits in some convenient coordinates, correct up to local topological equivalence, for parameter values \((\beta_1, \beta_2)\) near the curves \(F_1\) and \(F_2\) (for \(\beta_1 < \frac{1}{4}(\beta_2)^2, \beta_1 = \frac{1}{4}(\beta_2)^2, \beta_1 > \frac{1}{4}(\beta_2)^2\); with \(\beta_2 > 0\) fixed, or with \(\beta_2 < 0\) fixed).

(c) Verify from part (a) that the curve, emanating from the origin in the \((\beta_1, \beta_2)\)-plane, \(H: \beta_1 = 0, \beta_2 < 0\), corresponds to one of two equilibria that exist for (2.1) being a nonhyperbolic “linear centre” or “Hopf point” (purely imaginary eigenvalues for its linearization). Then for fixed \(\beta_2 < 0\), and considering \(\beta_1\) as a bifurcation parameter, use Theorem 3.6 to show that a Hopf bifurcation for (2.1) occurs on the curve \(H\), as the parameter \(\beta_1\) increases through the bifurcation value \(0\). Determine the local dynamics up to local topological equivalence, including existence and stability of limit cycles for parameter values on one side or the other of the curve \(H\) (determine which side). Draw three representative two-dimensional local phase portraits in some convenient coordinates near the critical equilibrium, correct up to local topological equivalence, for parameter values \((\beta_1, \beta_2)\) near the curve \(H\) (for \(\beta_1 < 0, \beta_1 = 0, \beta_1 > 0\); with \(\beta_2 < 0\) fixed). Also draw global phase portraits in the original \((x_1, x_2)\) coordinates for \((\beta_1, \beta_2)\) near \(H\), including
both equilibria. Observe that for \((\beta_1, \beta_2)\) just to the left of the Hopf bifurcation curve \(H\), and for \((\beta_1, \beta_2)\) just to the left of the fold bifurcation curve \(F_1\), the phase portraits are not locally topologically equivalent. This implies there should be (at least) another bifurcation, not yet determined, in the open region between \(H\) and \(F_1\).

3. In (2.1) of the previous question, assume \(\beta_1 < (\beta_2)^2/4\), let \(p_0^0(\beta) = (x_0^0, 0) \in \mathbb{R}^2\) denote the “left” equilibrium from part (a) of the previous question (whose linearization has a positive determinant), and make the coordinate shift

\[
y = p_0^0(\beta) + u
\]

in \(\mathbb{R}^2\) to obtain

\[
\frac{du_1}{dt} = u_2, \quad \frac{du_2}{dt} = -\alpha^2 u_1 + x_1^0 u_2 + u_1^2 - u_1 u_2,
\]

where \(\alpha > 0\) is defined by

\[
\alpha^2 = \sqrt{(\beta_2)^2 - 4\beta_1}.
\]

Then make the rescalings

\[
u_1 = \alpha^2 v_1, \quad u_2 = \alpha^3 v_2, \quad x_1^0 = \alpha^2 \gamma, \quad t = \frac{s}{\alpha},
\]

where \(\gamma\) is a constant, to get

\[
\frac{dv_1}{ds} = v_2, \quad \frac{dv_2}{ds} = -v_1 + \frac{v_2^2}{\alpha} + \alpha(\gamma v_2 - v_1 v_2).
\]

(3.1)

In the limit \(\alpha \to 0\) this system becomes Hamiltonian (but because of the time rescaling, this limit is singular with respect to the original system).

(a) Set \(\alpha = 0\) in the system (3.1), find a suitable Hamiltonian function and sketch the global phase portrait. Then verify that for \(\alpha = 0\),

\[
(v_1^0(s), v_2^0(s)) = \left(1 - \frac{3}{2} \text{sech}^2 \left(\frac{s}{2}\right), \frac{3}{2} \text{sech}^2 \left(\frac{s}{2}\right) \tanh \left(\frac{s}{2}\right)\right)
\]

is a homoclinic orbit for (3.1).

(b) Use Melnikov’s method to show that for all sufficiently small \(\alpha > 0\), there exists a homoclinic orbit for (3.1) if \(\gamma = \gamma^{SL}(\alpha)\), a unique value given by the Implicit Function Theorem, where

\[
\gamma^{SL}(\alpha) = \gamma_0 + O(|\alpha|),
\]

and the homoclinic orbit breaks when \(\gamma \neq \gamma^{SL}(\alpha)\). Find an explicit numerical value for the leading-order constant \(\gamma_0\).

(c) Transform \(\gamma = \gamma^{SL}(\alpha)\) back into the original parameters \(\beta_1\) and \(\beta_2\) of (2.1), to obtain a homoclinic (or saddle loop) bifurcation curve \(SL\) for (2.1) in the form

\[
\beta_1 = \beta_1^{SL}(\beta_2) = C(\beta_2)^2 + O(|\beta_2|^4).
\]

Find explicit values for the leading-order coefficient \(C \neq 0\) and higher-order power \(q > 2\).

(d) It can be proved for (2.1) that there is never more than one cycle, and that there are no more bifurcations (local or global) other than those on the four curves already found. Given these facts, draw the four curves \(F_1, F_2, H, SL\) on the same diagram in the \((\beta_1, \beta_2)\)-plane. Use Theorem 4.1 to help draw three 2-dimensional phase portraits for (2.1), in \((x_1, x_2)\) coordinates, for parameter values \((\beta_1, \beta_2)\) near the curve \(SL\) (for \(\beta_1 < \beta_1^{SL}(\beta_2)\), \(\beta_1 = \beta_1^{SL}(\beta_2)\), \(\beta_1 > \beta_1^{SL}(\beta_2)\); with \(\beta_2\) fixed).