1. Consider the one-parameter family of one-dimensional maps

\[ x \mapsto \alpha x(1 - x), \quad x \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}^1. \]

Globally these are not diffeomorphisms, but locally they can be. We can always consider forward orbits \( \{x_k\}_{k=0}^{\infty} \), and in some intervals we can consider both forward and backward orbits.

(a) Show that for all \( 0 < \alpha < 4 \), the closed interval \([0, 1]\) is forward invariant.

(b) For \( 0 < \alpha < 4 \), find all branches of fixed points \( x_0^{(j)}(\alpha) \) in the closed interval \([0, 1]\), and determine the linearized stability of each fixed point. Summarize your results in a branching diagram (use a horizontal \( \alpha \)-axis with \( 0 < \alpha < 4 \), and a vertical \( x \)-axis with \( 0 \leq x \leq 1 \)). From your branching diagram, identify what kind of bifurcation occurs at \( \alpha = 1 \).

(c) Show that a flip bifurcation occurs at \( \alpha = 3 \). Determine the existence and stability of cycles of period 2, at least for \( \alpha \) sufficiently near 3. Summarize your results by redrawing the branching diagram of part (b) and then adding the new information.

2. Poincaré normal forms can be developed for maps, and families of maps. We consider in parts (b) and (c) a useful example in one state space dimension.

(a) Consider a map \( x \mapsto f(x) \), where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism. Show that under a coordinate change \( x = h(y) \), where \( h : \mathbb{R}^n \to \mathbb{R}^n \) is a homeomorphism, the map \( x \mapsto f(x) \) is transformed into the [topologically equivalent] map \( y \mapsto g(y) \), where \( g(y) = h^{-1}(f(h(y))) \), i.e. \( g = h^{-1} \circ f \circ h \) (Hint: It may be easier to use the recursion notation for maps \( x_{k+1} = f(x_k), k \in \mathbb{Z} \).)

(b) Now let \( n = 1, x \in \mathbb{R}^1, y \in \mathbb{R}^1 \). Show that there exists a smooth coordinate change

\[ x = h(y) = y + \delta_0 y^2, \]

where \( h : \mathbb{R}^1 \to \mathbb{R}^1 \) is a local diffeomorphism, that transforms the map

\[ x \mapsto f(x) = \mu_0 x + b x^2 + c x^3 + O(|x|^4) \]

where \( \mu_0 = -1 \) and \( b, c \) are real constants, into the [locally topologically equivalent] map [i.e. the Poincaré normal form up to order 3]

\[ y \mapsto \mu_0 y + (b^2 + c) y^3 + O(|y|^4). \]

First, determine a Taylor series expansion for the inverse \( y = h^{-1}(x) \) up to a sufficiently high order. Then, show how to choose \( \delta_0 \) so that the coefficient of \( y^2 \) in the transformed map vanishes. Finally, determine the resulting coefficient of \( y^3 \) in the transformed map.

(c) Continuing from part (b) with \( n = 1, x \in \mathbb{R}^1, y \in \mathbb{R}^1 \), let \( \alpha \) belong to an open interval that contains 0, and show that there exists a family of smooth coordinate changes

\[ x = h(y, \alpha) = y + \delta(\alpha) y^2 \]
3. Consider the ODE
\[ x \mapsto \mu(\alpha) x + f_2(\alpha) x^2 + f_3(\alpha) x^3 + O(|x|^4), \]
where \( \mu(\alpha) = -1 - a\alpha + O(|\alpha|^2), \) \( f_2(\alpha) = b + O(|\alpha|), \) \( f_3(\alpha) = c + O(|\alpha|) \) are all smooth for \( \alpha \) in an open interval that contains 0, into the [locally topologically equivalent] family of maps [i.e. the Poincaré normal form up to order 3]
\[ y \mapsto \mu(\alpha) y + g_3(\alpha) y^3 + O(|y|^4), \]
where \( g_3(0) = b^2 + c. \) Show how to choose \( \delta(\alpha) \) so that the coefficient of \( y^2 \) in the transformed family vanishes for all \( \alpha, \) and then determine the coefficient of \( y^3 \) in the transformed family. \( \text{(Hint: For } \alpha = 0 \text{ this should reduce to part (b).)} \) Thus to leading order we have
\[ y \mapsto -y - a\alpha y + (b^2 + c) y^3 + O(|\alpha|^2|y| + |\alpha||y|^3 + |y|^4). \]
\[ (1) \]

(d) Calculate the second iterate of (1) up to leading order terms, with remainder \( O(|\alpha|^2|y| + |\alpha||y|^3 + |y|^4). \) [Thus, one can use Poincaré normal forms to simplify the family before taking the second iterate. This makes finding the second iterate much easier.]

3. Consider the ODE
\[ \dot{x} = f(x) = Ax + f^{(2)}(x) + O(|x|^3), \quad x = (x_1, x_2)^\top \in \mathbb{R}^2, \]
where \( 0 \in \mathbb{R}^2 \) is an equilibrium, \( f^{(2)} \in H_2, \) and
\[ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

(a) Find a \( 6 \times 6 \) matrix representation \( M^{(2)}_A \) of \( L_A : H_2 \to H_2 \) and use it to determine the range \( L_A(H_2) \) of \( L_A. \) Show that there exists a coordinate change
\[ x = h(y) = y + h^{(2)}(y), \quad h^{(2)} \in H_2, \]
that “removes” all but two of the six coefficients at order 2, that transforms (2) into a Poincaré normal form up to order 2,
\[ \dot{y} = Ay + r^{(2)}(y) + O(|y|^3) \]
given by
\[ \dot{y}_1 = y_2 + O(||(y_1, y_2)||^3), \quad \dot{y}_2 = ay_1^2 + by_1y_2 + O(||(y_1, y_2)||^3), \]
\[ (3) \]

where \( a, b \) are constants (“normal form coefficients”).

(b) Find explicitly an \( h^{(2)} \in H_2 \) that has the property specified in part (a), and find the normal form coefficients \( a \) and \( b \) in (3), in terms of second-order partial derivatives of the components of the original vector field \( f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))^\top, \) evaluated at the equilibrium \((0, 0)\).
4. The FitzHugh-Nagumo family of ODEs

\[
\begin{align*}
\dot{U} &= \gamma + U + V - \frac{1}{3}U^3, \\
\dot{V} &= \rho(\delta - U - \beta V), \\
\end{align*}
\]  

(4)

is a simplified model of electrical conduction along a nerve axon. Here \(\gamma, \rho\) and \(\delta\) are real constants (parameters).

(a) Make the change of variables

\[
\begin{align*}
U &= x_1, \\
V &= \frac{\delta}{\beta} + x_2, \\
\gamma &= \alpha - \frac{\delta}{\beta} \\
\end{align*}
\]

to transform the family (4) into

\[
\begin{align*}
\dot{x}_1 &= \alpha + x_1 + x_2 - \frac{1}{3}x_1^3, \\
\dot{x}_2 &= -\rho(x_1 + \beta x_2). \\
\end{align*}
\]  

(5)

Let \(0 < \rho < 1\) and \(0 < \beta < 1\) be fixed. Treating \(\alpha\) as a bifurcation parameter, show that for each \(\alpha \in \mathbb{R}^1\) there exists a unique equilibrium \((x_1, x_2) = (x_1^0(\alpha), x_2^0(\alpha))\). (Hint: Parametrize the family of equilibria by \(x_1 \in \mathbb{R}^1\), as \(\alpha = \phi(x_1), \ x_2 = \psi(x_1)\), then show that \(\phi : \mathbb{R}^1 \to \mathbb{R}^1\) is a global diffeomorphism and therefore the inverse \(x_1 = \phi^{-1}(\alpha)\) exists and is smooth for all \(\alpha \in \mathbb{R}^1\).) An explicit formula for \(x_1^0(\alpha)\) is not required.

(b) For each \(\alpha \in \mathbb{R}^1\), determine the linearized stability of the equilibrium for (5). In particular, show that the linearization has purely imaginary eigenvalues for two distinct finite values \(\alpha_1 < \alpha_2\). Determine the stability of \((x_1^0(\alpha), x_2^0(\alpha))\) for \(\alpha < \alpha_1\), for \(\alpha_1 < \alpha < \alpha_2\), and for \(\alpha_2 < \alpha\).

(c) Analyze each of the two Hopf bifurcations in the family (5), to determine the existence and stability of limit cycles, for \(\alpha\) sufficiently near \(\alpha_1\) and for \(\alpha\) sufficiently near \(\alpha_2\). Assume that \(0 < \rho < 1\), \(0 < \beta < 1\), and \(\rho\beta^2 - 2\beta + 1 \neq 0\). Draw schematic branching diagrams (Auto-style), with \(\alpha\) as the bifurcation parameter.