1. Consider the one-parameter family of one-dimensional maps

\[ x \mapsto \alpha x (1 - x), \quad x \in \mathbb{R}, \quad \alpha \in \mathbb{R}. \]  

(4.1.1)

Globally these are not diffeomorphisms, but locally they can be. We can always consider forward orbits \( \{x_k\}_{k=0}^\infty \), and sometimes we can consider both forward and backward orbits.

(a) Show that for all \( 0 < \alpha < 4 \), the closed interval \([0, 1]\) is forward invariant under (4.1.1).

(b) For \( 0 < \alpha < 4 \), find explicitly all branches of fixed points \( p_0[j](\alpha) \) for (4.1.1) in the closed interval \([0, 1]\), and determine the linearized stability of each fixed point. Summarize your results in a branching diagram (use a horizontal \( \alpha \)-axis with \( 0 < \alpha < 4 \), and a vertical \( x \)-axis with \( 0 \leq x \leq 1 \)). Just from your branching diagram, identify what kind of bifurcation occurs at \( \alpha = 1 \) (no further verification is required).

(c) Show that a flip bifurcation for (4.1.1) occurs at \( \alpha = 3 \). Determine the existence and stability of cycles of period 2, at least for \( \alpha \) sufficiently near 3. Summarize your results by redrawing the branching diagram of part (b) and then adding the new information.

2. Poincaré normal forms can be developed for maps, and families of maps. We consider in parts (b) and (c) a useful example in one state space dimension.

(a) Consider a map \( x \mapsto f(x) \), where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism. Show that under a coordinate change \( x = h(y) \), where \( h : \mathbb{R}^n \to \mathbb{R}^n \) is a homeomorphism, the map \( x \mapsto f(x) \) transforms into the [topologically equivalent] map \( y \mapsto g(y) \), where \( g(y) = h^{-1}(f(h(y))) \), i.e. \( g = h^{-1} \circ f \circ h \) (Hint: It may be easier to use the recursion notation for maps \( x_{k+1} = f(x_k), k \in \mathbb{Z} \).)

(b) Now let \( n = 1, u \in \mathbb{R}, v \in \mathbb{R} \). Show that there exists a smooth coordinate change of the form

\[ u = h_0(v) = v + h^{(2)}_0 v^2, \]  

(4.2.1)

where \( h^{(2)}_0 \) is a constant and \( h_0 : \mathbb{R} \to \mathbb{R} \) is a local diffeomorphism (near \( v = 0 \)), that transforms any map of the form

\[ u \mapsto \hat{f}(u) = \mu_0 u + \hat{f}_{20} u^2 + \hat{f}_{30} u^3 + O(|u|^4), \]

where \( \mu_0 = -1 \) and \( \hat{f}_{20}, \hat{f}_{30} \) are constants, into the map [i.e. the Poincaré normal form up to order 3]

\[ v \mapsto \mu_0 v + g_{30} v^3 + O(|v|^4). \]  

(4.2.2)

where

\[ g_{30} = \hat{f}_{30} + \left( \frac{\hat{f}_{20}}{2} \right)^2. \]

First, find the Taylor series expansion for the inverse \( v = h_0^{-1}(u) \) up to a sufficiently high order. Then, show how to choose the coefficient \( h^{(2)}_0 \) of \( v^2 \) in the coordinate change (4.2.1) so that the coefficient \( g_{20} \) of \( v^2 \) vanishes in the transformed map \( v \mapsto \mu_0 v + g_{20} v^2 + g_{30} v^3 + O(|v|^4) \) (i.e. so that \( g_{20} = 0 \)). Finally, determine the resulting coefficient \( g_{30} \) of \( v^3 \) in the transformed map if this specific \( h^{(2)}_0 \) is used.
(c) As in part (b) let \( n = 1, u \in \mathbb{R}^1, v \in \mathbb{R}^1 \), now also let \( \alpha \) belong to an open interval that contains \( \alpha_0 \). For simplicity, assume \( \alpha_0 = 0 \). Show that there exists a family of coordinate changes of the form

\[
u = h(v, \alpha) = v + h^{(2)}(\alpha) v^2\]

where \( h^{(2)}(\alpha) \) is a smooth real-valued function of \( \alpha \) in an open interval that contains \( 0 \), and for each \( \alpha \) near \( 0, h(\cdot, \alpha) : \mathbb{R}^1 \to \mathbb{R}^1 \) is a local diffeomorphism, that transforms any family of maps of the form

\[
u \mapsto \mu(\alpha) u + f_2(\alpha) u^2 + f_3(\alpha) u^3 + O(|u|^4),
\]

where \( \mu(\alpha) = -1 + O(|\alpha|), f_2(\alpha) = f_{20} + O(|\alpha|), f_3(\alpha) = f_{30} + O(|\alpha|) \) are all smooth real-valued functions of \( \alpha \) in an open interval that contains \( 0 \), into the family of maps

\[
u \mapsto \mu(\alpha) v + g_3(\alpha) v^3 + O(|v|^4),
\]

where

\[g_3(0) = f_{30} + \left( f_{20} \right)^2.\]

Show how to choose the coefficient \( h^{(2)}(\alpha) \) of \( v^2 \) in the family of coordinate changes so that the coefficient of \( v^2 \) vanishes in the transformed family of maps for all \( \alpha \) near \( 0 \). Finally, determine the resulting coefficient of \( v^3 \) in the transformed family of maps if this specific \( h^{(2)}(\alpha) \) is used. (Hint: For \( \alpha = 0 \) this reduces to part (b).) If \( \mu(\alpha) = -1 - a\alpha + O(|\alpha|^2), \) then to leading order we have

\[
u \mapsto -v - a\alpha v - b v^3 + O(|\alpha|^2|v| + |\alpha||v|^3 + |v|^4), \tag{4.2.3}
\]

where \( -b = f_{30} + \left( f_{20} \right)^2. \)

(d) Calculate the second iterate of (4.2.3) up to leading order terms, with remainder \( O(|\alpha|^2|v| + |\alpha||v|^3 + |v|^4) \).

3. Consider the two-dimensional vector field \( \dot{x} = f(x) = Ax + f^{(2)}(x) + O(\|x\|^3), x = (x_1, x_2)^T \in \mathbb{R}^2 \), where

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f^{(2)} \in H_2. \tag{4.3.1}
\]

Note that the origin \((0, 0)\) is an equilibrium.

(a) Find a \((6 \times 6)\) matrix representation \( M^{(2)}_A \), of \( L_A : H_2 \to H_2 \), and determine a basis for the range \( L_A(H_2) \), of \( L_A \) in \( H_2 \). Show that there exists a coordinate change

\[
x = y + h^{(2)}(y), \quad h^{(2)} \in H_2,
\]

that transforms (4.3.1) into a Poincaré normal form up to order 2,

\[
\dot{y} = Ay + v^{(2)}(y) + O(\|y\|^3),
\]

given specifically by

\[
\dot{y}_1 = y_2 + O(\|(y_1, y_2)\|^3), \quad \dot{y}_2 = a y_1^2 + b y_1 y_2 + O(\|(y_1, y_2)\|^3), \tag{4.3.2}
\]

where \( a \) and \( b \) are constants, thus “removing” all but two of the six second-order coefficients,
(b) Find explicitly an $h^{(2)}$ that accomplishes the transformation shown to exist in part (a), and find the normal form coefficients $a$ and $b$ of (4.3.2), in terms of partial derivatives of components of the original vector field

$$f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix},$$

evaluated at the equilibrium $(0, 0)$. Give the complementary subspace $\tilde{H}_2$, to $L_A(H_2)$ in $H_2$, that corresponds to the Poincaré normal form (4.3.2), and check that it is indeed complementary.

4. The FitzHugh-Nagumo family of vector fields

$$\dot{U} = \gamma + U + V - \frac{1}{3}U^3, \quad \dot{V} = \rho(\delta - U - \beta V), \quad (4.4.1)$$
is a simplified model of electrical conduction along a nerve axon. Here $\gamma$, $\rho$ and $\delta$ are real constants (parameters).

(a) Make the change of variables

$$U = x_1, \quad V = \frac{\delta}{\beta} + x_2, \quad \gamma = \alpha - \frac{\delta}{\beta}$$
to transform the family (4.4.1) into

$$\dot{x}_1 = \alpha + x_1 + x_2 - \frac{1}{3}x_1^3, \quad \dot{x}_2 = -\rho(x_1 + \beta x_2). \quad (4.4.2)$$

Let $0 < \rho < 1$ and $0 < \beta < 1$ be fixed. Treating $\alpha$ as a bifurcation parameter, show that for each $\alpha \in \mathbb{R}^1$ there exists a unique equilibrium $(x_1, x_2) = p_0^\alpha(\alpha) = (p_0^\alpha_1(\alpha), p_0^\alpha_2(\alpha))$. (Hint: Parametrize the family of equilibria by $x_1 \in \mathbb{R}^1$, as $x_1 = \phi(x_1)$, $x_2 = \psi(x_1)$, then show that $\phi : \mathbb{R}^1 \to \mathbb{R}^1$ is a global diffeomorphism and therefore the inverse $x_1 = \phi^{-1}(\alpha)$ exists and is smooth for all $\alpha \in \mathbb{R}^1$.) An explicit formula for $p_0^\alpha_1(\alpha) = \phi^{-1}(\alpha)$ is not required.

(b) For each $\alpha \in \mathbb{R}^1$, determine the linearized stability of the equilibrium for (4.4.2). In particular, show that the linearization has purely imaginary eigenvalues for two distinct values $\alpha_1 < \alpha_2$. Determine the stability of $p_0^\alpha(\alpha)$ for $\alpha < \alpha_1$, for $\alpha_1 < \alpha < \alpha_2$, and for $\alpha_2 < \alpha$.

(c) Analyze each of the two Hopf bifurcations in the family (4.4.2), to determine the existence and stability of limit cycles, for $\alpha$ sufficiently near $\alpha_1$ and for $\alpha$ sufficiently near $\alpha_2$. Assume that $0 < \rho < 1$, $0 < \beta < 1$, and $\rho \beta^2 - 2\beta + 1 \neq 0$. Draw schematic branching diagrams (Auto-style), with $\alpha$ as the bifurcation parameter on the horizontal axis.