1. The Lotka-Volterra model, that describes interacting (nondimensionalized) populations of a prey species $x_1(t)$ and a predator species $x_2(t)$, is given by

$$\dot{x}_1 = x_1(1 - x_2), \quad \dot{x}_2 = \delta(x_1 - 1)x_2,$$

in the closed quarter-plane $\overline{Q} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, where $\delta$ is a positive constant (proportional to the per capita death rate of the predator population when there is no prey; changing its value has no qualitative effect on the phase portraits).

(a) Find all equilibria for (3.1.1) in the closed quarter-plane $\overline{Q}$, and for each equilibrium determine its linearized stability.

(b) Show that the nonnegative coordinate axes $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 = 0\}$ and $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \geq 0\}$ are both invariant sets for (3.1.1), and determine the global dynamics restricted to these two invariant sets. Draw one-dimensional phase portraits in the two invariant sets.

(c) Let $L(x_1, x_2) = \delta x_1 + x_2 - \ln(x_1^2x_2)$, and calculate $\dot{L}(x_1, x_2)$ in the open quarter-plane $Q = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. Using $L$ (locally) as a Lyapunov function, determine the stability of the equilibrium in $Q$. (Hint: Use the second derivative test on $L(x_1, x_2)$ at the equilibrium.)

(d) Show that the vector field is not a Hamiltonian vector field, but that in the open quarter-plane $Q$, it is orbitally equivalent to a Hamiltonian vector field. Sketch the two-dimensional global phase portrait of the system. [The global phase portrait is well-known and can be found in many undergraduate textbooks. Orbitally equivalent vector fields have the same phase portraits, but along their orbits they have different parametrizations by the “time” variable. One problem with this model is that the vector field is not structurally stable: there exist arbitrarily small changes to the vector field that give topologically non-equivalent flows. So arbitrarily small changes to the model can give qualitatively different predictions.]

2. The following family of vector fields models a (nondimensionalized) planar pendulum subject to constant damping $\delta \geq 0$ and constant applied torque at the pivot $\gamma \geq 0$:

$$\dot{\varphi} = \nu, \quad \dot{\nu} = \gamma - \delta \nu - \sin(\varphi), \quad x = (\varphi, \nu) \in S^1 \times \mathbb{R}^1 = X. \quad (3.2.1)$$

(a) Find all equilibria for (3.2.1) in the cylinder $X = S^1 \times \mathbb{R}^1$. For each equilibrium in $X$, perform a linearized stability analysis and then determine the (actual) stability of the equilibrium as far as can be determined from the linearized stability analysis (if not possible, say so). Consider the cases

i. $\delta = 0, \gamma > 0$ (there are further subcases here);

ii. $\gamma = 0, 0 < \delta \ll 1$.

In some cases/subcases, there are precisely two distinct equilibria $p_{[1]}^0 = (\varphi_{[1]}^0, \nu_{[1]}^0)$ and $p_{[2]}^0 = (\varphi_{[2]}^0, \nu_{[2]}^0)$ in the cylinder $X$, one of which, label it $p_{[2]}^0$, is a hyperbolic saddle.

(b) Sketch global phase portraits in the cylinder $X = S^1 \times \mathbb{R}^1$ (indicate explicitly what values of $\varphi$ are being identified) for the cases/subcases in part (b) where there are
precisely two distinct equilibria, one of which is a hyperbolic saddle. Clearly indicate the *global* stable and unstable manifolds \( W^s(p_{[2]}^0) \) and \( W^u(p_{[2]}^0) \) of the hyperbolic saddle equilibrium. Justify your phase portraits as much as possible using analytical methods, such as linearized stability analysis for *local* information (when it is possible to use Theorem 2.2), and analysis of a Hamiltonian or Lyapunov function.

3. **Transcritical bifurcations in one-dimensional vector fields.** Let \( f : \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) be \( C^3 \), and consider the one-parameter family of one-dimensional vector fields

\[
\frac{dx}{dt} = f(x, \alpha). \tag{3.3.1}
\]

Assume \( f \) satisfies a *constraint*

\[
f(0, \alpha) = 0 \quad \text{for all } \alpha \quad \text{(TC.0.i)}
\]

that belong to some open interval of \( \alpha \)-values that contains some special value \( \alpha_0 \). Thus there is an equilibrium \( p_{[1]}^0(\alpha) = 0 \) for all \( \alpha \) in some open interval that contains \( \alpha_0 \), i.e. \( p_{[1]}^0(\alpha) \equiv 0 \) is a smooth branch of equilibria. Next, assume that a *bifurcation* condition

\[
f_x(0, \alpha_0) = 0 \quad \text{(TC.0.ii)}
\]

holds, so that \( p_{[1]}^0(\alpha_0) = 0 \) is a nonhyperbolic equilibrium for \( \alpha = \alpha_0 \). Further, assume that a *transversality* condition

\[
a = f_{x\alpha}(0, \alpha_0) \neq 0, \quad \text{(TC.1)}
\]

and a *nondegeneracy* condition

\[
b = \frac{1}{2} f_{xx}(0, \alpha_0) \neq 0 \quad \text{(TC.2)}
\]

both hold.

(a) Show that because of the constraint (TC.0.i), we can write \( f(x, \alpha) = x\tilde{f}(x, \alpha) \) for some function \( \tilde{f} \) that is smooth enough near \( (0, \alpha_0) \) (you do not need to worry about the degree of smoothness).

(b) Show that there is another smooth branch of equilibria \( p_{[2]}^0(\alpha) \) defined for \( \alpha \) in an open interval containing \( \alpha_0 \), that crosses the first branch \( p_{[1]}^0(\alpha) \equiv 0 \) transversally at \( \alpha = \alpha_0 \) in the \((\alpha, x)\)-plane, i.e. \( (p_{[2]}^0)'(\alpha_0) \neq 0 \). Draw the four possible branching diagrams for (3.3.1), depending on the signs of \( a \) and \( b \), indicating stability of equilibria in the conventional way. *Hint:* Use the result of part (a), then use the Implicit Function Theorem. [This local bifurcation is called *transcritical*. The results can be summarized by saying that (3.3.1) locally topologically equivalent, at \((0, \alpha_0)\), to the normal form \( \frac{dy}{dx} = a\beta y + by^2 \), at \((0, 0)\), and to one of the two topological normal forms \( \frac{d\eta}{d\tau} = \beta \eta \pm \eta^2 \), at \((0, 0)\).]

(c) Show that transcritical bifurcations are *not* structurally stable, by giving a simple *explicit* example of a smooth function \( f_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^1 \) so that there exist \( \varepsilon \) arbitrarily close to 0 such that the perturbed family

\[
\dot{x} = \alpha x - x^2 + \varepsilon f_1(x, \alpha, \varepsilon)
\]

has no transcritical bifurcation near \((x, \alpha) = (0, 0)\). Draw the topologically non-equivalent branching diagrams \((\alpha \ vs. \ x)\) for \( \varepsilon = 0 \) and for \( \varepsilon \neq 0 \).
(d) (Transcritical bifurcations, generalized.) Suppose the family has a smooth branch of equilibria \( x = p^0_{[i]}(\alpha) \) (not necessarily always zero), so that (TC.0.i) is replaced by

\[
f(p^0_{[i]}(\alpha), \alpha) = 0 \quad \text{for all } \alpha, \quad \text{(TC.0.i')}
\]

and (TC.0.ii) is replaced by

\[
f_x(p^0_{[i]}(\alpha_0), \alpha_0) = 0. \quad \text{(TC.0.ii')}
\]

Give the appropriate replacements for (TC.1) and (TC.2) in terms of partial derivatives of \( f \) evaluated at \( (p^0_{[i]}(\alpha_0), \alpha_0) \), so that there is a transcritical bifurcation for (3.3.1).

**Hint:** let \( x = p^0_{[i]}(\alpha) + u \) and work with the corresponding family of differential equations for \( u \).

4. Let \((x_1, x_2)\) denote the usual rectangular coordinates in the plane \( \mathbb{R}^2 \), and consider the one-parameter family

\[
\dot{x}_1 = \alpha ax_1 - \omega x_2 + b(x_1^2 + x_1 x_2), \quad \dot{x}_2 = \omega x_1 + \alpha ax_2 + b(x_1^2 x_2 + x_2^3), \quad \text{(3.4.1)}
\]

where \( \omega > 0 \), \( a \neq 0 \), \( b \neq 0 \) are fixed real constants, and \( \alpha \in \mathbb{R}^1 \) is a bifurcation parameter.

(a) Observe that the origin \((0, 0)\) is an equilibrium for (3.4.1). Determine the linearized stability of this equilibrium. There are different cases.

(b) Transform the family into complex coordinates \( z = x_1 + ix_2 \) for \((x_1, x_2) \in \mathbb{R}^2 \) (complexify, then “realify”) and also into polar coordinates \( r \in (0, \infty), \theta \in \mathbb{S}^1 \) (where \( x_1 = r \cos(\theta), \ x_2 = r \sin(\theta) \)), for \((x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}\). In the cases from part (a) where linearized stability analysis fails to determine stability of the equilibrium at the origin, now determine the stability of that equilibrium (as unstable, Lyapunov stable but not stable, or stable). There are different cases. Notice in some cases there are limit cycles. In these cases, also determine the stability of the limit cycles (with reference to the polar coordinate representation is sufficient, explicit computation of Floquet multipliers is not required).

(c) Fix the signs of \( a \) and \( b \) (four different cases), then (use the polar coordinate representation to help) sketch the phase portraits of (3.4.1) in the \((x_1, x_2)\)-plane, for the different \( \alpha \)-values that give topologically non-equivalent phase portraits.