1. Suppose \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is \( C^p \), with \( p \geq 1 \). By the Fundamental Theorem of Calculus, the solution \( x(t) = \varphi(t, t_0, x_0, \alpha), t \in \mathcal{J}(t_0, x_0, \alpha) \), of the initial value problem

\[
\dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0,
\]

satisfies the equivalent integral equation

\[
\varphi(t, t_0, x_0, \alpha) = x_0 + \int_{t_0}^{t} f(s, \varphi(s, t_0, x_0, \alpha), \alpha) \, ds. \tag{1}
\]

By Theorem 2.1, \( \varphi(t, t_0, x_0, \alpha) \) is differentiable with respect to the initial value \( x_0 \) and also with respect to the parameter \( \alpha \). By differentiating (1) with respect to \( x_0 \) or \( \alpha \), and then with respect to \( t \) (assuming that one may interchange the orders of differentiation and integration), one can derive some useful facts about the derivatives (matrices) \( \varphi_{x_0} \) and \( \varphi_{\alpha} \).

(a) Let \( \Phi(t) = \varphi_{x_0}(t, t_0, x_0, \alpha) \), and determine the initial value problem satisfied by the \( n \times n \) matrix \( \Phi(t) \) (the derivative of the solution with respect to the initial value).

(b) Let \( \Theta(t) = \varphi_{\alpha}(t, t_0, x_0, \alpha) \), and determine the initial value problem satisfied by the \( n \times m \) matrix \( \Theta(t) \) (the derivative of the solution with respect to the parameter).

2. Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is \( C^p \), \( p \geq 1 \). Let \( x(t) = \varphi(t, t_0, x_0) \) be the unique solution of the initial value problem for the autonomous system

\[
\dot{x} = f(x), \quad x(t_0) = x_0,
\]

starting at initial value \( x_0 \) at arbitrary initial time \( t_0 \), with \( \varphi(t, t_0, x_0) \) defined for \( t \) belonging to the maximal open interval of existence \( \mathcal{J}(t_0, x_0) \).

(a) Let \( y(t) = \varphi(t, 0, x_0), t \in \mathcal{J}(0, x_0) \), be the unique solution of

\[
\dot{y} = f(y), \quad y(0) = x_0,
\]

starting at the same initial value \( x_0 \) but at initial time 0. Let \( z(t) = y(t - t_0) \), and show that \( x(t) = z(t) \), i.e. \( \varphi(t, t_0, x_0) = \varphi(t - t_0, 0, x_0) \), and \( \mathcal{J}(t_0, x_0) = t_0 + \mathcal{J}(0, x_0) \). So for an autonomous system, without loss of generality in the initial value problem we may assume that the initial time is \( t_0 = 0 \).

(b) Prove that \( \varphi(t + s, 0, x_0) = \varphi(t, 0, \varphi(s, 0, x_0)) \), i.e. \( \varphi^{t+s}(x_0) = \varphi^t \circ \varphi^s(x_0) \), for all \( t, s \in \mathbb{R}, x_0 \in \mathbb{R}^n \) such that both sides are defined.

3. Consider the initial value problem for the one-parameter family of autonomous vector fields in \( \mathbb{R}^1 \), with \( \alpha \in \mathbb{R}^1 \),

\[
\dot{x} = \alpha x - x^2, \quad x(0) = x_0 \in \mathbb{R}^1. \tag{2}
\]

(a) Sketch phase portraits in the state space (i.e. the \( x \)-axis) \( \mathbb{R}^1 \), for each of the three cases \( \alpha < 0, \alpha = 0, \alpha > 0 \). Note that you do not have to “solve” the initial value problems explicitly, in order to sketch the phase portraits.
4. Let \( \alpha \) be a constant.

(b) Sketch the phase portrait, in the \( \alpha x \)-plane, of the autonomous vector field in \( \mathbb{R}^2 \),

\[
\dot{\alpha} = 0, \quad \dot{x} = \alpha x - x^2.
\]

Use a horizontal \( \alpha \)-axis and a vertical \( x \)-axis.

(c) In a separate sketch, draw curves in the \( \alpha x \)-plane showing the locations of all equilibria \( x^0 = x^0(\alpha) \). Use a solid curve to denote a “branch” of (asymptotically) stable equilibria, and a dashed curve to denote a branch of unstable equilibria.

(d) For \( \alpha = 0 \) only:

i. by elementary methods, find explicitly the solution of the initial value problem (2) i.e. the local flow \( \varphi^t(x_0) = \varphi(t, 0, x_0, 0) \) including its maximal open interval of existence \( \mathcal{J}(0, x_0, 0) \) (you may need to consider different cases of \( x_0 \) separately);

ii. find explicit numerical values of \( t, s, \) and \( x_0 \) such that only one of \( \varphi^{t+s}(x_0) = \varphi(t+s, 0, x_0, 0) \) or \( \varphi^{s} \circ \varphi^{s}(x_0) = \varphi(t, 0, \varphi(s, 0, x_0, 0), 0) \) is defined, but the other is not.

4. Let \((x_1, x_2)\) denote rectangular coordinates in the plane \( \mathbb{R}^2 \), and consider the vector field

\[
\dot{x}_1 = x_1 - x_2 + x_1^2 - x_1x_2, \quad \dot{x}_2 = x_1 + x_2 + x_1x_2 - x_1^2x_2 - x_2^3. \tag{3}
\]

(a) Observe that the origin \((0, 0)\) is an equilibrium. Determine the linearized stability of this equilibrium.

(b) Complexify the vector field (3) by letting \((x_1, x_2) \in \mathbb{C}^2\). In \( \mathbb{C}^2 \), make the change of coordinates

\[
\begin{align*}
z_1 &= x_1 + ix_2, & z_2 &= x_1 - ix_2,
\end{align*}
\]

and write the vector field (3) transformed into the coordinates \((z_1, z_2)\) (use the chain rule, etc.).

(c) Now “realify” the vector field in part (b) by restricting it to the invariant subspace \( \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = \bar{z}_1, \} \), where \( \bar{z}_1 \) is the complex conjugate of \( z_1 \). Polar coordinates in the plane \( r \in \mathbb{R}_+ = (0, \infty), \theta \in \mathbb{S}^1 \) may be expressed as \( z_1 = re^{i\theta}, z_2 = \bar{z}_1 = re^{-i\theta} \) or \( r^2 = z_1\bar{z}_2 = z_1\bar{z}_1, 2i\theta = \log z_1 - \log z_2 = \log z_1 - \log \bar{z}_1 \). Write the vector field (3) transformed into polar coordinates \((r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^1\).

(d) Show that there are two circles \( r = a \) and \( r = b \), with \( 0 < a < b \), such that \( \dot{r} > 0 \) on \( r = a \) and \( \dot{r} < 0 \) on \( r = b \). Thus the compact (closed and bounded) annular region \( \mathcal{A} = \{(x_1, x_2) \in \mathbb{R}^2 : a^2 \leq x_1^2 + x_2^2 \leq b^2 \} \) is positively invariant (or “trapping”) and it contains no equilibria.

(e) Let \( \Sigma \) be the positive \( x_1 \)-axis

\[\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 = 0\} \]

Show that \( \Sigma \) is a cross-section to the vector field (3), and also carefully show that any initial value on \( \Sigma \) gives a solution to the initial value problem that always returns to \( \Sigma \) after a positive finite time. Thus for this vector field, the domain of the Poincaré map \( P \) is all of \( \Sigma \), \( P \) is a “global” Poincaré map.

(f) Carefully show that \( P(a) > a \) and \( P(b) < b \), where \( a \) and \( b \) are as in part (d). Use the Intermediate Value Theorem for continuous functions in \( \mathbb{R} \) (look it up in a calculus textbook or online, apply it correctly) to show that the Poincaré map \( P \) has at least one fixed point in \( \Sigma \), and therefore the vector field (3) has at least one periodic orbit in \( \mathbb{R}^2 \). (If you know the Poincaré-Bendixson Theorem, it can be applied to give the same conclusion.)