1. Suppose $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is $C^p$, with $p \geq 1$. By the Fundamental Theorem of Calculus, the solution $x(t) = \varphi(t, t_0, x_0, \alpha)$, $t \in J(t_0, x_0, \alpha)$, of the family of initial value problems

$$\dot{x} = f(t, x, \alpha), \quad x(t_0) = x_0,$$

satisfies the equivalent integral equation

$$\varphi(t, t_0, x_0, \alpha) = x_0 + \int_{t_0}^{t} f(s, \varphi(s, t_0, x_0, \alpha), \alpha) \, ds.$$

By Theorem 2.1, $\varphi(t, t_0, x_0, \alpha)$ is differentiable with respect to the initial value $x_0 \in \mathbb{R}^n$ and also with respect to the parameter $\alpha \in \mathbb{R}^m$. By differentiating (1) with respect to $x_0$ or with respect to $\alpha$, and then with respect to $t$ (assuming that one is allowed to interchange the orders of differentiation and integration), one can derive some useful facts about the partial derivatives $\varphi_{x_0}$ and $\varphi_{\alpha}$ of the solution.

(a) If $\Phi(t) = \varphi_{x_0}(t, t_0, x_0, \alpha)$, determine the initial value problem satisfied by the $n \times n$ matrix function $\Phi(t)$.

(b) If $\Theta(t) = \varphi_{\alpha}(t, t_0, x_0, \alpha)$, determine the initial value problem satisfied by the $n \times m$ matrix function $\Theta(t)$.

2. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^p$, $p \geq 1$. Let $x(t) = \varphi(t, 0, x_0)$, $t \in J(0, x_0)$, be the unique solution of the initial value problem for the autonomous system

$$\dot{x} = f(x), \quad x(0) = x_0,$$

starting at initial value $x_0$ at initial time 0, with $\varphi(t, 0, x_0)$ defined for $t$ belonging to the (unique) maximal open interval of existence $J(0, x_0)$.

(a) Let $y(t) = \varphi(t, t_0, x_0)$, $t \in J(t_0, x_0)$, be the unique solution of

$$\dot{y} = f(y), \quad y(t_0) = x_0,$$

starting at the same initial value $x_0$ as (2.3) but at an arbitrary initial time $t_0$. Let $z(t) = x(t - t_0)$, and show that $y(t) = z(t)$, i.e. $\varphi(t, t_0, x_0) = \varphi(t - t_0, 0, x_0)$, and that $J(t_0, x_0) = t_0 + J(0, x_0)$.

(b) Prove that $\varphi(s + t, 0, x_0) = \varphi(s, 0, \varphi(t, 0, x_0))$, i.e. $\varphi(s + t)(x_0) = \varphi^s \circ \varphi^t(x_0)$, for all $s, t \in \mathbb{R}, x_0 \in \mathbb{R}^n$ such that both sides are defined.

3. Consider the initial value problem for the family of vector fields in $\mathbb{R}^1$, with $\alpha \in \mathbb{R}^1$,

$$\dot{x} = \alpha x - x^2, \quad x(0) = x_0 \in \mathbb{R}^1.$$ 

(a) Sketch phase portraits in the state space (i.e. the $x$-axis) $\mathbb{R}^1$, for each of the three cases $\alpha < 0$, $\alpha = 0$, $\alpha > 0$. You do not have to “solve” the initial value problems explicitly, in order to sketch the phase portraits.
(b) Sketch the phase portrait, in the \((\alpha, x)\)-plane, of the vector field in \(\mathbb{R}^2\),

\[
\dot{\alpha} = 0, \quad \dot{x} = \alpha x - x^2.
\]

Use a horizontal \(\alpha\)-axis and a vertical \(x\)-axis.

(c) In a separate sketch, draw curves in the \((\alpha, x)\)-plane showing the locations of all equilibria \(p^0 = r^0|\alpha\). Use a solid curve to denote a “branch” of stable equilibria, and a dashed curve to denote a branch of unstable equilibria.

(d) For \(\alpha = 0\) only:

i. by elementary methods, find explicitly the solution of the initial value problem (2)

\[
\text{i.e. the local flow } \varphi(x_0) = \varphi(t, 0, x_0, 0) \text{ including the maximal open interval of existence } J(0, x_0, 0) \text{ (you should consider different cases of } x_0 \text{ separately)};
\]

ii. find explicit numerical values of \(s, t, \text{ and } x_0\) such that only one of

\[
\varphi^{s+t}(x_0) = \varphi(s+t, 0, x_0, 0) \text{ or } \varphi^s \circ \varphi^t(x_0) = \varphi(s, 0, \varphi(t, 0, x_0, 0), 0) \text{ is defined, but the other is not.}
\]

4. Let \((x_1, x_2)\) denote rectangular coordinates in the plane \(\mathbb{R}^2\), and consider the vector field

\[
x'_1 = x_1 - x_2 + x_1^2 - x_1 x_2^2, \quad \dot{x}_2 = x_1 + x_2 + x_1 x_2 - x_1^2 - x_2^2.
\] (3)

(a) Observe that the origin \((0, 0) \in \mathbb{R}^2\) is an equilibrium. Determine the linearized stability of this equilibrium.

(b) Complexify the vector field (3) by letting \((x_1, x_2) \in \mathbb{C}^2\). In \(\mathbb{C}^2\), make the change of coordinates

\[
z_1 = x_1 + ix_2, \quad z_2 = x_1 - ix_2,
\]

and write the vector field (3) transformed into the coordinates \((z_1, z_2)\) (use the Chain Rule, etc.).

(c) Now “realify” the vector field in part (b) by restricting it to the invariant subspace

\[
\{ (z_1, z_2) \in \mathbb{C}^2 : z_2 = \bar{z}_1, \}
\]

where \(\bar{z}_1\) is the complex conjugate of \(z_1\). Polar coordinates in the plane \(r \in \mathbb{R}_+ = (0, \infty), \theta \in S^1\) may be expressed as \(z_1 = re^{i\theta}, \bar{z}_1 = re^{-i\theta}\) or \(r^2 = z_1 \bar{z}_2 = z_1 \bar{z}_1, 2\theta = \log z_1 - \log z_2 = \log z_1 - \log \bar{z}_1\). Write the vector field (3) (for \((x_1, x_2) \neq (0, 0)\) transformed into polar coordinates \((r, \theta) \in \mathbb{R}_+ \times S^1\).

(d) Show that there exist two circles \(r = a\) and \(r = b\), with \(0 < a < b\), such that \(\dot{r} > 0\) on \(r = a\) and \(\dot{r} < 0\) on \(r = b\). Thus, the compact (closed and bounded) annular region

\[
\{ (x_1, x_2) \in \mathbb{R}^2 : a^2 \leq x_1^2 + x_2^2 \leq b^2 \}
\]

is positively invariant (or “trapping”) and it contains no equilibria.

(e) Let \(\Sigma\) be the positive \(x_1\)-axis

\[
\Sigma = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 = 0 \}.
\]

Show that \(\Sigma\) is a cross-section to the vector field (3), and also carefully show that any initial value on \(\Sigma\) gives a solution to the initial value problem that exists and always returns to \(\Sigma\) after a positive finite time. For this vector field, the domain of the Poincaré map \(P\) is all of \(\Sigma\), and we may say \(P\) is a “global” Poincaré map.

(f) Carefully show that \(P(a) > a\) and \(P(b) < b\), where \(a\) and \(b\) are as in part (d). Use

the Intermediate Value Theorem for continuous functions in \(\mathbb{R}\) (look it up in a calculus textbook or online, apply it correctly) to show that the Poincaré map \(P\) has at least one fixed point in \(\Sigma\), and therefore the vector field (3) has at least one periodic orbit in \(\mathbb{R}^2\). (If you know the Poincaré-Bendixson Theorem, it can be applied to give the same conclusion.)