1. Let $A$, $B$ and $Q$ be real $n \times n$ matrices.

(a) Show that if $AB = BA$, then $e^{A}e^{B} = e^{A+B}$. Explain precisely where $AB = BA$ is used. (Consequently, $e^{As}e^{At} = e^{A(s+t)}$ for all real numbers $s$ and $t$, stated as property (DS.1) in the first lecture.)

(b) Show that if $Q$ is nonsingular, then $e^{Q^{-1}AQ} = Q^{-1}e^{A}Q$. (Consequently, we can compute the exponential of $A$ if we can find the exponential of its real normal form $R$.)

(c) Show that if

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}$$

is block diagonal with square blocks $A_1$ and $A_2$ (here $O$ denotes a rectangular block of zeros), then its exponential $e^{At}$, $t \in \mathbb{R}$, is also block diagonal,

$$e^{At} = \begin{pmatrix} e^{A_1t} & O \\ O & e^{A_2t} \end{pmatrix}.$$ 

(Consequently, if an initial value $x_0$ belongs to the subspace $V_j$ that corresponds to one of the square blocks then the solution to the initial value problem $x(t) = e^{At}x_0$ belongs to $V_j$ for all $t \in \mathbb{R}$, i.e. $V_j$ is an invariant subspace. From part (b), these subspaces remain invariant under a change of basis transformation.)

2. For each of the given real matrices $A$ below: $i)$ find the exponential $e^{At}$ for all $t \in \mathbb{R}$ explicitly; $ii)$ calculate $\lim_{t \to +\infty} e^{At}$ and directly (from the limit) determine whether the equilibrium $x = 0$ for the linear vector field $\dot{x} = Ax$ is: unstable, Lyapunov stable but not asymptotically stable, or asymptotically stable. (Hints: Use results from question 1. Consider all the different cases that could affect stability, e.g. nine cases in part (a): $\lambda_1$ negative, positive or zero and $\lambda_2$ negative, positive or zero.)

(a)

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

(b)

$$A = \begin{pmatrix} \mu_1 - \omega_1 & 0 & 0 & 0 & 0 \\ \omega_1 & \mu_1 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 - \omega_2 & 1 & 0 \\ 0 & 0 & \omega_2 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 & -\omega_2 \\ 0 & 0 & 0 & 0 & \omega_2 \end{pmatrix},$$

where $\mu_1 + i\omega_1, \mu_2 + \omega_2 \in \mathbb{C}$, $\mu_1, \mu_2, \omega_1, \omega_2$ all real, with $\omega_1, \omega_2 \neq 0$. 
3. For each of the given real matrices $B$ below: i) find the power $B^k$ for all $k \in \mathbb{Z}$ explicitly; ii) calculate $\lim_{k \to +\infty} B^k$ and directly (from the limit) determine whether the fixed point $x = 0$ for the linear diffeomorphism $x \mapsto Bx$ is: unstable, Lyapunov stable but not asymptotically stable, or asymptotically stable. (Hints: Carefully apply the binomial theorem for $(S + N)^k$, $k \geq 2$, to get a formula for $B^k$ at least for $k \geq 2$ but check your formula explicitly for $k = 1, 0, -1$ and show that it works for $k \leq -2$. Consider all the different cases that affect stability.)

(a) 

$$B = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 1 \\ 0 & 0 & \mu_2 \end{pmatrix},$$

where $\mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$.

(b) 

$$B = \begin{pmatrix} \rho_1 \cos \phi_1 & -\rho_1 \sin \phi_1 \\ \rho_1 \sin \phi_1 & \rho_1 \cos \phi_1 \end{pmatrix},$$

where $\rho_1 e^{i\phi_1} \in \mathbb{C} \setminus \{0\}$ ($\rho_1 > 0, \phi_1 \in S^1$). Describe what a typical orbit looks like (you may find complex coordinates or polar coordinates useful).

(c) 

$$B = \begin{pmatrix} \rho_1 \cos \phi_1 & -\rho_1 \sin \phi_1 & 1 & 0 \\ \rho_1 \sin \phi_1 & \rho_1 \cos \phi_1 & 0 & 1 \\ 0 & 0 & \rho_1 \cos \phi_1 & -\rho_1 \sin \phi_1 \\ 0 & 0 & \rho_1 \sin \phi_1 & \rho_1 \cos \phi_1 \end{pmatrix},$$

where $\rho_1 e^{i\phi_1} \in \mathbb{C} \setminus \{0\}$ ($\rho_1 > 0, \phi_1 \in S^1$).

4. Classify all 2-dimensional linear diffeomorphisms $x \mapsto Bx$, where $B$ is a real $2 \times 2$ matrix, $\Delta = \det B \neq 0$ ($\Delta > 0$ is case $+$, $\Delta < 0$ is case $-$), according to whether the fixed point $x = 0$ is $[i\pm]$ a hyperbolic orientation preserving (+) or orientation reversing (−) sink (both multipliers inside the unit circle); $[ii\pm]$ a hyperbolic orientation preserving/reversing source (both multipliers outside the unit circle); $[iii\pm]$ a hyperbolic orientation preserving/reversing saddle (one multiplier inside the unit circle and one outside); or $[iv\pm]$ non-hyperbolic (at least one multiplier on the unit circle):

(a) Plot a diagram in the $\sigma\Delta$-plane that shows the topological classification into the eight cases (four for $\Delta > 0$, four for $\Delta < 0$) above, where $\sigma = \text{tr} B$ is the trace of $B$. Use a horizontal $\sigma$-axis and vertical $\Delta$-axis. (Hints: First find the curves that correspond to $[iv\pm]$, since these form the boundaries of the open regions corresponding to cases $[i\pm]$, $[ii\pm]$ and $[iii\pm]$. Show that the “stability regions” corresponding to $[i\pm]$ are the open regions $\Delta < 1$, $\Delta > -1 - \sigma$, $\Delta > -1 + \sigma$ with $\Delta > 0$ or $< 0$.)

(b) For the nonhyperbolic cases $[iv\pm]$, find all possible subcases of real normal forms of $B$, and for each nonhyperbolic subcase (there are several), determine whether the fixed point $x = 0$ is: unstable, Lyapunov stable but not asymptotically stable, or asymptotically stable. Phase portraits are not required.