Bijections

Let $X$ and $Y$ be sets, and let $U \subseteq X$ be a subset. A (single-valued) map or function $f : X \to Y$ takes each element $x \in U$ to a unique element $f(x) \in Y$. We write

$$y = f(x), \quad \text{or} \quad x \mapsto f(x).$$

The domain (of definition) of $f$ is $U$, and the range or image of $f$ is the subset $f(U)$ of $Y$ given by

$$f(U) = \{ y \in Y : \text{there exists } x \in U \text{ such that } y = f(x) \}.$$

A map $f : X \to Y$ is one-to-one or is invertible or is an injection if for all $x_1, x_2 \in U$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$ (equivalently, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$). If $f$ is one-to-one, then there is an inverse map or inverse function $f^{-1} : Y \to X$ whose domain is $f(U)$ (notice that $f^{-1}$ is single-valued).

A map $f : X \to Y$ is onto (or is a surjection) if $f(U) = Y$.

A map $f : X \to Y$ is a (global) bijection (or one-to-one correspondence) if $U = X$ and $f$ is one-to-one and onto. If $f$ is a bijection, then its inverse map $f^{-1} : Y \to X$ is defined on all of $Y$.

Homeomorphisms

Let $X$ and $Y$ be topological spaces. Examples of topological spaces are: i) $\mathbb{R}^n$; ii) a metric space; iii) a smooth manifold (see below); iv) an open subset of a topological space. A map $f : X \to Y$ is a (global) homeomorphism if it is a (global) bijection, and both $f$ and its inverse map $f^{-1}$ are continuous maps.

Two topological spaces are equivalent (as topological spaces) if there is a homeomorphism from one topological space onto the other.

If $U \subseteq X$ and $V \subseteq Y$ are open subsets, and $f : U \to V$ is a homeomorphism, then we call $f$ a local homeomorphism from $X$ into $Y$. In this case, for brevity we usually write $f : X \to Y$ even when $f$ is actually a local homeomorphism $f : U \to V$ from $X$ into $Y$, and we call it a homeomorphism even if its domain $U$ is not all of $X$ or its range $V$ is not all of $Y$.

Manifolds, tangent spaces

We only consider manifolds that are smooth. The definition of a smooth manifold given here is sufficient for this course, but there are other, more general definitions.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, with $m < n$. Then the subset of $\mathbb{R}^n$ given by

$$X = \{ x \in \mathbb{R}^n : F(x) = 0 \}$$

(the solution set of $m$ equations in $n$ unknowns) is a smooth (or differentiable) manifold in $\mathbb{R}^n$ if the $m \times n$ derivative (matrix) $F_x(x)$ has rank $m$, the maximum possible, at each point $x \in X$. In this case the dimension of $X$ is $d = n - m$. If the map $F$ is $C^p$, then we say $X$ is a $C^p$ manifold.

Examples of smooth manifolds are: i) $\mathbb{R}^n$; ii) an open subset $U \subseteq \mathbb{R}^n$; iii) the graph $\{(x,y) : y = V(x)\}$ of a smooth function $V : \mathbb{R}^n \to \mathbb{R}^m$ defined on an open subset $U \subseteq \mathbb{R}^n$ (as an exercise, show that this satisfies the above definition for a manifold in $\mathbb{R}^{n+m}$ of dimension $n$); iii) the (1-dimensional) unit circle
$S^1 = \mathbb{R}^1/(2\pi\mathbb{Z}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 - 1 = 0\}; iv) \text{ the (2-dimensional) cylinder } S^1 \times \mathbb{R}^1; v) \text{ the } 2\text{-dimensional torus or 2-torus } T^2 = S^1 \times S^1; vi) \text{ an open subset of a manifold.}

A one-dimensional smooth manifold in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a smooth curve, a two-dimensional smooth manifold in $\mathbb{R}^3$ is a smooth surface.

If $X$ is a smooth manifold in $\mathbb{R}^n$ and $x \in X$, then the tangent space to $X$ at $x$ is the vector space $T_x X$ in $\mathbb{R}^n$ of all velocity vectors $v = \dot{\gamma}(0)$ for smooth curves $\gamma : \mathbb{R} \to X$ that lie in $X$ with $\gamma(0) = x$ ($v$ is the velocity vector pointing in the direction $\dot{\gamma}(0)$ from the point $x \in X$; think of it as a vector whose “tail” is at the point $x$). Alternatively, $T_x X$ is the vector space which is the orthogonal complement of $\{\nabla F_1(x), \cdots, \nabla F_m(x)\}$, where $F_j$ is the $j$th component of $F$, i.e.

$$T_x X = \{\nabla F_1(x), \cdots, \nabla F_m(x)\}^\perp.$$

The dimension of the tangent space $T_x X$ at $x$ is equal to the dimension of the smooth manifold $X$ (for any $x \in X$).

Diffeomorphisms

Let $X$ and $Y$ be smooth manifolds. A map $f : X \to Y$ is a (global) diffeomorphism if it is a (global) homeomorphism, and both $f$ and its inverse map $f^{-1}$ are differentiable (this implies that the manifolds $X$ and $Y$ must have the same dimension). The map $f$ is a $C^p$ diffeomorphism ($p \geq 1$) if $X$ and $Y$ are both $C^p$ manifolds, and both $f$ and $f^{-1}$ are $C^p$ maps.

If $U \subseteq X$ and $V \subseteq Y$ are open subsets of smooth manifolds, and $f : U \to V$ is a ($C^p$) diffeomorphism, then call $f$ a local ($C^p$) diffeomorphism from $X$ into $Y$. As for homeomorphisms, we usually write $f : X \to Y$ even when $f$ is actually a local diffeomorphism $f : U \to V$ from $X$ into $Y$ and we call it a diffeomorphism, even if its domain $U$ is not all of $X$ or its range $V$ is not all of $Y$. (If we really mean $f : X \to Y$ with domain $X$ and range $Y$, sometimes, for emphasis, we say $f$ is a global diffeomorphism.)

Exercise. Show that $f : \mathbb{R}^1 \to \mathbb{R}^1, f(x) = x^3$, is a global homeomorphism, but not a global diffeomorphism.