1. \[ \frac{dN}{dt} = rN(1 - \frac{N}{K}) - EN \]

(a) Non-negative equilibria, for \(0 < E < r\):

Solve for \(N\):

\[ O = rN(1 - \frac{N}{K}) - EN \]

\[ O = N \left[ r(1 - \frac{N}{K}) - E \right] \]

\[ N^* = 0 \text{ or } r(1 - \frac{N}{K}) - E = 0 \]

\[ N^* = K(1 - \frac{E}{r}) > 0 \text{ for } 0 < E < r \]

Linearized stability:

\[ f(N) = rN - \frac{rK}{K} N^2 - EN, \quad f'(N) = r - E - \frac{2rK}{K} N \]

i. \( A + N^* = 0 \), \( \lambda = f'(0) = r - E > 0 \text{ for } 0 < E < r \)

\( N^* = 0 \) is unstable

ii. \( A + N^* = K(1 - \frac{E}{r}) \), \( \lambda = f'(K - \frac{rE}{r}) = E - r < 0 \text{ for } 0 < E < r \)

\( N^* = K(1 - \frac{E}{r}) \) is stable

(b) Sustainable yield \( Y = E \cdot K(1 - \frac{E}{r}) \)

(c) \( Y = KE - \frac{K}{r} E^2 \), find absolute maximum value:

\[ \frac{dY}{dE} = K - 2 \frac{K}{r} E = 0 \text{ if } E = \frac{r}{2}, \text{ a critical point} \]

1st derivative test

\[ \frac{dY}{dE} = K(1 - \frac{3}{2} E) = \frac{2K}{r} (\frac{r}{2} - E) > 0 \text{ if } E < \frac{r}{2}; \quad Y(E) \text{ is increasing} \]

\[ \frac{dY}{dE} < 0 \text{ if } E > \frac{r}{2}; \quad Y(E) \text{ is decreasing} \]

By the "First Derivative Test for Absolute Extreme Values" (Stewart, p. 32E),

\[ Y_m = Y(\frac{r}{2}) = K \frac{r}{2} - \frac{K}{r} (\frac{r}{2})^2 = \frac{Kr}{4} \]

is the absolute maximum value of \( Y \), with \( E_m = \frac{r}{2} \).

(d) If \( E \) is slightly above \( E_m \), then \( N^* \) is slightly less than \( K(1 - \frac{E}{r}) \), still positive and stable. The fish population \( N \) will be near this stable equilibrium (there is no danger of extinction), and the sustainable yield \( Y = EK(1 - \frac{E}{r}) \) will be slightly below \( Y_m = \frac{Kr}{4} \).
2. \( \frac{dn}{dt} = \frac{Gp_n}{Gn + t} - kn \)

(a) Put in dimension less form

Let \( x = \frac{n}{A} \), \( t = \frac{t}{B} \) where \( A \) has the same units as \( n \), \( B \) has the same units as \( t \)

Then \( n = Ax \), \( \frac{dn}{dt} = \frac{1}{B} \frac{dx}{dt} \) and

\[ \frac{dx}{dt} = \frac{Gp_Ax}{GAx + t} - kAx \]

\[ \frac{dx}{dt} = \frac{Gp_Bx}{GAx + t} - kBx \]

\[ \frac{dx}{dt} = \frac{Gp_Bx}{GAx + t} - kBx \]

Choose \( A = \frac{t}{f} \), \( B = \frac{1}{k} \), \( \mu = \frac{Gp_B}{f} \) so

\( x = \frac{Gn}{f} \), \( t = kt \), \( \mu = \frac{Gp}{fk} \)

(b) Nonnegative equilibria: solve for \( x \) in

\( 0 = \frac{\mu x}{x+1} - x \)

\( 0 = x \left( \frac{\mu}{x+1} - 1 \right) \quad x^* = 0 \) or \( \frac{\mu}{x+1} = 1 \), \( \mu = x+1 \), \( x^* = \mu - 1 \)

Nonnegative equilibria are

\( x^* = 0 \) for all \( \mu \), \( x^* = \mu - 1 \) for \( \mu > 1 \)

Linearized stability:

\[ f'(x) = \frac{\mu (x+1) - (x)(1)}{(x+1)^2} - 1 = \frac{\mu}{(x+1)^2} - 1 \]

At \( x^* = 0 \), \( \lambda = f'(0) = \mu - 1 \)

\( x^* = 0 \) is stable if \( \mu < 1 \), unstable if \( \mu > 1 \)

At \( x^* = \mu - 1 \), \( \lambda = f'(\mu-1) = \frac{1}{\mu} - 1 = \frac{1 - \mu}{\mu} \), \( \mu > 1 \)

\( x^* = \mu - 1 \) is stable for \( \mu > 1 \) (the only values we care about)

(c) Bifurcation diagram

\[
\begin{array}{c}
\text{Transcritical bifurcation} \\
\text{at } \mu = 1, x = 0 \\
\end{array}
\]

(d) Graph of Rate in \( = \frac{\mu x}{x+1} \) is same as Michaelis-Menten rate

\[ y = \frac{\mu x}{x+1} \]

slope of tangent at \( 0 \) is \( \mu \)

Graph of Rate out \( = x \) is easy
How these graphs interact for $x > 0$ depends on $\mu$:

$\mu < 1$

Rate in < Rate out for all $x > 0$
Rate in - Rate out < 0 for all $x > 0$
Phase portrait

$\mu > 1$

Rate in - Rate out > 0
Rate in - Rate out < 0
Phase portrait

Bifurcation diagram with phase portraits superimposed

There is no hysteresis. As $\mu$ increases past 1 the laser switches on continuously without jumping, then if $\mu$ decreases below 1 the laser switches off continuously without jumping, and at the same value as switching on.