MATH 360
Midterm Test

Marks

Allowed aid: one 8.5" × 11" sheet (both sides) of notes.

1. In the fisheries model

\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - EN, \quad 0 \leq N < \infty,
\]

\(N(t)\) is the population of fish at time \(t\), and \(r\) (intrinsic per capita growth rate), \(K\) (carrying capacity), \(E\) (fishing effort) are all positive constants.

(a) For \(0 < E < r\), find (analytically) all nonnegative equilibria \(N^*\), and determine the linearized stability of each nonnegative equilibrium.

(b) Find the sustainable yield \(Y = EN^*\) as a function of \(E\) for \(0 < E < r\), where \(N^*\) is a stable equilibrium population.

(c) Find the maximum sustainable yield \(Y_m\), the absolute maximum value of \(Y\) as a function of \(E\) for \(0 < E < r\). Carefully justify that your expression for \(Y_m\) is the absolute maximum value. What value of \(E = E_m\) produces \(Y_m\)?

(d) Explain what this model predicts will happen to the fish population \(N\) and the sustainable yield \(Y\) if the fishing effort \(E\) is slightly above \(E_m\).

2. In the laser model

\[
\frac{dn}{dt} = \frac{Gpn}{Gn + f} - kn, \quad 0 \leq n < \infty,
\]

\(n(t)\) is the number of photons in the solid state laser device at time \(t\), while \(f, G, k\) are all positive constants, and \(p\) is a real constant (i.e. it could be positive, negative or zero).

(a) Show that the equation can be written in dimensionless form as

\[
\frac{dx}{d\tau} = \frac{\mu x}{x + 1} - x, \quad 0 \leq x < \infty, \tag{1}
\]

where you must find the definitions of \(x, \tau, \mu\) in terms of \(n, t, f, G, k, p\).

(b) Find (analytically) all nonnegative equilibria \(x^*\) for (1), and determine the linearized stability of each nonnegative equilibrium.

(c) Draw the bifurcation diagram for (1), showing all nonnegative equilibria and their stabilities for all real \(\mu\). Classify (graphically) all local bifurcations.

(d) Find all nonnegative equilibria \(x^*\) for (1) graphically, by expressing (1) as a one-box model

\[
\frac{dx}{d\tau} = Rate\ in - Rate\ out, \quad where \quad Rate\ in = \frac{\mu x}{x + 1}, \quad Rate\ out = x,
\]

graphing both \(Rate\ in\) and \(Rate\ out\) as functions of \(x\) on the same plot, and interpreting your plot. Sketch phase portraits in the \(x\)-axis for representative values of \(\mu\) (for \(\mu\) in intervals or at specific values, as appropriate), then redraw the bifurcation diagram with the phase portraits superimposed. Does hysteresis occur? Explain briefly.