Example 5.3  Simple harmonic oscillator

Spring model (Hooke's Law) \[ F = -kx \]  \[ x = \text{displacement} \]
\[ m = \text{mass} \]

Newton's 2nd law  \[ ma = F \]
\[ a = \text{acceleration} \]
\[ \frac{dv}{dt} = \frac{dx}{dt} \]
\[ v = \text{velocity} \]
\[ \frac{dv}{dt} = \frac{d^2x}{dt^2} \]
\[ m\frac{d^2x}{dt^2} = -kx \]

2nd order eqn.

Can treat 1st order system. Define \[ \frac{dx}{dt} = v \]  \[ \text{(velocity of mass) } \]

Then \[ \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\frac{k}{m}x \]
so equivalent system is

\[ \begin{cases}
\frac{dx}{dt} = v \\
\frac{dv}{dt} = -\frac{k}{m}x
\end{cases} \]

Look for solution \((x(t), v(t))\) in \(xv\) plane

E.g. \(m = 1\), \(k = 4\)
\[ \begin{cases}
\frac{dx}{dt} = v \\
\frac{dv}{dt} = -4x
\end{cases} \]

Nullclines \[ \frac{dx}{dt} = 0 \iff v = 0 \]  \[ \text{(x-nulls)} \]
\[ \frac{dv}{dt} = 0 \iff x = 0 \]  \[ \text{(v-nulls)} \]

Finding

\[ \text{Superimpose} \]

But hard to guess phase portrait.

More information from conservation of energy.

\[ \text{Kinetic energy } \frac{1}{2}mv^2 \]
\[ \text{Potential energy } \frac{1}{2}kx^2 \]
\[ E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \] is constant i.e. \( \frac{dE}{dt} = 0 \)

Proof.
\[
\frac{dE}{dt} = \frac{1}{2} m 2v \frac{dv}{dt} + \frac{1}{2} k 2x \frac{dx}{dt} = m v \ddot{x} + k x v = 0 \text{ for all } x, v
\]

This means \((x(t), v(t))\) have to follow curves \( E = \text{constant} \)

\[ v^2 + 4 = 1 \]

\[ E = \frac{1}{2} v^2 + \frac{1}{2} 4 x^2 = \frac{1}{2} v^2 + 2 x^2 \]

If \( E \) is a constant \( \frac{E}{2} = v^2 + \frac{x^2}{4} \) (what kind of curve?)

\[ E = 2 \]

\[ v^2 + \frac{x^2}{4} = 1 \]

\[ E = 8 \]

\[ v^2 + \frac{x^2}{4} = 4 \]

\[ -2 \]

\[ v^2 + \frac{x^2}{16} = 1 \]

For \( E > 0 \) these curves \( E = \frac{1}{2} v^2 + 2 x^2 \) are nested ellipses

\[ E = 0 \quad 0 = \frac{1}{2} v^2 + 2 x^2 \quad \text{or} \quad v = 0, x = 0 \quad \text{the origin} \]

Together with ant Whilst undifferentiated icircular
phase portrait is

Every solution is periodic in time
More justification of quasi-static approx. To 1st order
\[
\frac{d\hat{s}}{d\hat{c}} = -\hat{s} + \left( \frac{k_{e1}}{k_{1}s_0} + \hat{\tau} \right) \hat{c}
\]
\[
e \frac{d\hat{c}}{d\hat{t}} = \hat{s} - \left( \frac{k_{e1} + k_{e2}}{k_{1}s_0} + \hat{\tau} \right) \hat{c}, \quad e = \frac{e_0}{s_0} > 0
\]

Quasi-static approx. equivalent to putting e = 0. What if 0 < e << 1?
\[
e \frac{d\hat{c}}{d\hat{t}} = \frac{1}{e} \left[ \hat{s} - \left( \frac{k_{e1} + k_{e2}}{k_{1}s_0} \right) \hat{c} \right]
\]

This is very large.
\[
e \frac{d\hat{c}}{d\hat{t}} \text{ is very large, } \frac{d\hat{s}}{d\hat{t}} \text{ is not very large unless } \left[ \hat{s} - \left( \frac{k_{e1}}{k_{1}s_0} \right) \hat{c} \right] \text{ is very small}
\]

For arbitrary solids \((\dot{s}(t), \dot{c}(t))\) very quickly move to regime where \(\hat{s} - (x+\hat{c}) \hat{c} \approx 0\) and stop there.

Therefore \(\hat{s} - (x+\hat{c}) \hat{c} = 0\) is a good approx.

(quasi-static approx.)