Example 4.10: Selection adds to the Moran process with N individuals.

Let

\[ w_A = w \text{ fitness of type A} \]

\[ w_B = w \text{ fitness of type B} \]

If \( w_B > 1 \), relative fitness of type B

\[ \text{mean fitness} \quad W = W = \frac{w_A}{N} + \frac{w_B}{N} = \frac{w_A + w_B}{N} \]

Then there are \( j \) individuals of type A

\[ \text{prob. that A gives birth} = \frac{w_A}{N} = \frac{w_A}{w_B + w_A} \]

\[ \text{prob. that B gives birth} = \frac{w_B}{N} = \frac{w_B}{w_B + w_A} \]

\[ \text{prob. that A dies} = \frac{w_B}{N} \]

\[ \text{prob. that B dies} = \frac{w_A}{N} \]

Transition probabilities become

\[ p_{i,j} = \frac{N-j}{N} \cdot \frac{w_B}{w_B + w_A} \]

\[ p_{j,i} = \frac{N-j}{N} \cdot \frac{w_A}{w_B + w_A} \]

\[ p_{i,j} = 0 \quad \text{if } i \neq j, j \neq 1 \]

Same arguments as before give fixation probabilities

\[ (1-p_{i,j}) F_j = p_{i,j} F_{j-1} + \frac{N-j}{N} \cdot F_{j-1} \]

Now solve \( F_j \) gets complicated. And result is

\[ F_j = \frac{1}{1 - (\frac{w}{N})^j} \]

Consider with Moran process when \( j = 1 \): probability that 1 individual of type A

produces offspring and eventually take over the population.

\[ F_1 = \frac{1}{N} \quad \text{in Moran process (neutral drift, no selection)} \]

\[ \to 0 \quad \text{as } N \to \infty \]

\[ F_1 \approx 0 \quad \text{in large population} \]

\[ F_1 = \frac{1 - \frac{w}{N}}{1 - (\frac{w}{N})^N} \quad \text{with selection} \]

If A individuals are more fit than B individuals, \( w > 1 \)

\[ \text{In this case } (\frac{w}{N})^N \to 0 \quad \text{as } N \to \infty \]

\[ F_1 \to 1 - \frac{1}{w} \quad \text{as } N \to \infty \]

\[ F_1 \approx 1 - \frac{1}{w} \quad \text{in large population} \]

Even in a large population, there is a finite probability that one more fit

individual of type A could have descendants offspring that take over the entire

population. e.g., if \( w = 2 \) then \( F_1 \approx 1 - \frac{1}{2} = \frac{1}{2} \), a 50% chance!
5. INTRODUCTION TO CONTINUOUS-TIME DYNAMICAL SYSTEMS IN TWO DIMENSIONS

Example 5.1 Carbon cycle (very simplified) as 2-box model.

\[ M_1(t) = \text{amount (mass) of carbon in atmosphere (as CO}_2) \]
\[ M_2(t) = \text{amount (mass) of carbon in living beings (biomass)} \]

\[ \begin{array}{c}
\text{Box 1 - atmosphere} \\
M_1(t)
\end{array} \quad \begin{array}{c}
\text{Box 2 - biomass} \\
M_2(t)
\end{array} \]

\[ \begin{array}{c}
\text{photocycles} \\
\text{respiration} + \text{decomposition}
\end{array} \quad \begin{array}{c}
F_{12} \\
F_{21}
\end{array} \]

Assume rate of removal of carbon from each box is proportional to its mass.

\[ \text{Photosynthesis: } F_{12} = k_1 M_1 \quad \text{Gigatons per year (Gt y}^{-1}) \quad \text{constant } k_1 > 0 \]

\[ \text{Respiration and decomposition: } F_{21} = k_2 M_2 \quad \text{Gt y}^{-1} \quad \text{constant } k_2 > 0 \]

For box 1, \( \frac{dM_1}{dt} = \text{Rate in - Rate out} = F_{21} - F_{12} = k_2 M_2 - k_1 M_1 \)

For box 2, \( \frac{dM_2}{dt} = \text{Rate in - Rate out} = k_1 M_1 - k_2 M_2 \)

We get system of ODEs:

\[ \begin{align*}
\frac{dM_1}{dt} & = -k_1 M_1 + k_2 M_2 \\
\frac{dM_2}{dt} & = k_1 M_1 - k_2 M_2
\end{align*} \]

*E.g.* \( k_1 = 0.075 \text{ y}^{-1} \); \( k_2 = 0.1 \text{ y}^{-1} \)

Example 5.2 \[ \begin{align*}
\frac{dx}{dt} & = ax + by \\
\frac{dy}{dt} & = cx + dy
\end{align*} \]

### Linear system with constant coefficients.

Different predictions for different \( a, b, c, d \).

Typically, initial conditions \( x(0) = x_0, y(0) = y_0 \) are given.

For a specific instance, take \( a = 1, b = 12, c = 3, d = 1 \)

\[ \begin{align*}
\frac{dx}{dt} & = x + 12y \\
\frac{dy}{dt} & = 3x + y
\end{align*} \]

Think of solution \((x(t), y(t))\) as a point moving in the \(xy\) plane as \( t\) increases.

Think of \((x(t), y(t))\) as a point moving in the \(xy\) plane as \( t\) increases.

**Phase vector** is tangent to curve at \((x(t), y(t))\).
Useful to know nullclines: places where "phase velocity" \( \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \) is either purely vertical or purely horizontal.

Since
\[
\frac{dx}{dt} = x + 12y, \quad \frac{dy}{dt} = 0 \iff x + 12y = 0 \quad \text{(i.e., } y = -\frac{1}{12}x \text{)}
\]

Simultaneously
\[
\frac{dx}{dt} = 0 \iff 3x + y = 0 \quad \text{(i.e., } y = -3x \text{)}
\]

Savannah hares

Guess the movement to be \( (x(t), y(t)) \): phase portrait is an equilibrium.

Steady paths can't cross (violate theorem on uniqueness of solution).

There are special directions (find eigenvectors).

Next example: 
\[
2\ddot{x} + k\dot{x} + x = 0, \quad E = m_g + \frac{1}{2}k\dot{x}^2 = \text{constant}.
\]