Example 4.9 continued. Moran spheres

transition probabilities

\[ p_{i,j} = P(X_{n+1} = i \mid X_n = j) = \left( \frac{n_i}{N} \right)^{i-j} \left( \frac{n_j}{N} \right)^{j-i} \]

for \( i \neq j \)

stationary measure
\[ \pi_i = \frac{n_i}{N} \]

e.g. \( N = 5, n = 0 \)
\[ f_0^0 = 0, f_0^1 = 0, f_0^2 = 0, f_0^3 = 0, f_0^4 = 0 \]
\( f_0^0 = 0 \)

e.g. \( N = 5, n = 1 \)
\[ f_1^1 = 0, f_1^0 = 0, f_1^2 = 0.32, f_1^3 = 0.44, f_1^4 = 0.24 \]
\( f_1^1 = 0.0576, f_1^0 = 0.4846, f_1^2 = 0.3024, f_1^3 = 0.0256, f_1^4 = 0.0056 \)

The two states \( X_n = 0 \) and \( X_n = N \) are special. They are called absorbing states, since once \( X_n \)

reaches either value, it remains at that value for all later times.

\( X_n = 0 \) is extinctive, no individuals of type A remain, none can be born in the future.

\( X_n = N \) is extinctive, all individuals are type A, only type A can be born in the future.

\[ p_{i,0} = 1 \quad \text{and} \quad p_{i,0} = 0 \quad \text{for all } i \neq 0 \]
\[ p_{N,0} = 1 \quad \text{and} \quad p_{N,0} = 0 \quad \text{for all } i \neq N \]

e.g. \( N = 5, j = 3 \) from above.

\[ P(X_2 = 4) = p_{4,3} f_4^3 + p_{4,4} f_4^4 \]
\[ = \sum_{j=0}^{N-1} p_{i,j} f_j^i \]

For any \( N \),

\[ P(X_{n+1} = i) = f_i^{(n+1)} = \sum_{j=0}^{N-1} p_{i,j} f_j^i \quad \text{for } i = 0, 1, \ldots, N; \quad n = 0, 1, \ldots \]

This is the same formula as for a matrix times a (column) vector.

\[ e.g. \ N = 5 \]
\[ \begin{pmatrix} 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.16 & 0.24 & 0 & 0 & 0 \\ 0.08 & 0.16 & 0 & 0 & 0 \\ 0.04 & 0.08 & 0.16 & 0 & 0 \\ 0 & 0 & 0.04 & 0.08 & 0.16 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.16 & 0.24 & 0 & 0 & 0 \\ 0.08 & 0.16 & 0 & 0 & 0 \\ 0.04 & 0.08 & 0.16 & 0 & 0 \\ 0 & 0 & 0.04 & 0.08 & 0.16 \end{pmatrix} \]

Let \( \mathbf{f}_0 = \begin{pmatrix} f_0^0 \\ f_0^1 \\ f_0^2 \\ f_0^3 \\ f_0^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \), then \( \mathbf{f}_n = \mathbf{P}^n \mathbf{f}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \)
\[ \mathbb{P}^n = P^2 \mathbb{P}_{\mathbb{P}} \cdot P \mathbb{P} + \mathbb{P} \mathbb{P} + P \mathbb{P}^2 \]

In general, \( \mathbb{P}^n = P^n \mathbb{P} \) where \( P \) is the rate matrix of the Markov chain.

**Question:** If the process starts at \( x=0 \) in state \( j \) (with certainty), what is the fixation probability, the probability that \( X_n \) will reach \( N \) eventually?

For example, \( \mathbb{P}^0 \cdot F_0 = 0 \) (if \( j=0 \) then there is no chance of fixation)
\[ j=1 \text{ to } N: F_n = 1 \] (if \( j=N \) then fixation has already occurred)

If \( 0 < j < N \), there are three disjoint ways fixation could occur:
- \( j \) increases to \( j+1 \) and then \( j+1 \) leads to fixation
- \( j \) decreases to \( j-1 \) and then \( j-1 \) leads to fixation
- \( j \) remains at \( j \) and then \( j \) leads to fixation

- \( F_j = P_{j+1,j} F_{j+1} + P_{j-1,j} F_{j-1} + P_{j,j} F_j \)
- \( (1 - P_{j,j}) F_j = P_{j+1,j} F_{j+1} + P_{j-1,j} F_{j-1} \)

\[ F_j = \frac{P_{j+1,j}}{1 - P_{j,j}} F_{j+1} + \frac{P_{j-1,j}}{1 - P_{j,j}} F_{j-1} \]

**Note:**
\[ 1 = \left( \frac{j + N - 1}{N} \right)^2 = \frac{j^2 + 2j(N - j) + (N - j)^2}{N^2} = \left( \frac{j}{N} \right)^2 + \frac{2(N - j)}{N} \frac{j}{N} \]

- \( 1 - P_{j,j} = 2\left( \frac{j}{N} \right) \left( \frac{N-j}{N} \right) = 2 P_{j+1,j} = 2 P_{j-1,j} \)
- \( F_j = \frac{1}{2} F_{j+1} + \frac{1}{2} F_{j-1} \)

Caused update of this linear difference eqn. in \( F_j \) is \( F_j = \delta_j + \delta_{j+1} \) where \( \delta_j \) are arbitrary constants. Check this is a solution:
\[ \delta_j + \delta_{j+1} = \frac{1}{2} (\delta_{j} + \delta_{j+1}) + \frac{1}{2} (\delta_{j} + \delta_{j+2}) (j-1) \]

**Find the constants \( c, d \) from known values of \( F_j \):**
\[ 0 = F_0 = \delta_0 + d \cdot 0 \Rightarrow c = 0 \Rightarrow F_j = d_j \]
\[ 1 = F_N = d \cdot N \Rightarrow d = \frac{1}{N} \Rightarrow F_j \sim \frac{j}{N} \]

**Fixation probability is\( F_0 = \frac{d}{N} \)**

**E.g., if there are 5 individuals, of which 3 are type A, the probability that eventually all are type A is \( \frac{3}{5} \).**

**Cautious note:** even if there is no selection, one type could take over in a finite population just by chance.