\[ x_{n+1} = r x_n (1 - x_n), \quad 0 < x_n < 1, \quad 0 < r < 4 \]

- \( r = 3 \)

- \( r < 3 \) (region 3)

- \( r > 3 \) (region 3)

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**Numerical Experiments: Bifurcation Diagrams**

- initial condition \( x_0 \)
- compute \( x_n \), \( n = 1, 2, 3, \ldots \) up to some very large number \( N \)
- sort \( x_n \) values
- plot only \( x_n \) values, \( x_n \) values

- e.g., \( r = 0.25 \), \( x_0 = 0.575 \): \( x_1 = 0.375 \), \( x_2 = 0.125 \), \( x_3 = 0.5 \), \( x_4 = 0.25 \)
- \( r = 0.7 \), \( x_0 = 0.25 \): \( x_1 = 0.425 \), \( x_2 = 0.75 \), \( x_3 = 0.25 \), \( x_4 = 0.75 \)
- \( r = 0.9 \), \( x_0 = 0.5 \): \( x_1 = 0.625 \), \( x_2 = 0.375 \), \( x_3 = 0.625 \), \( x_4 = 0.375 \)

- change \( r \), do it again, e.g., \( r = 0.501 \)

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- \( r = 2.5 \)
- \( x_0 = 0.1 \)
- \( x_0 = 0.135 \)
- \( r = 0.5 \)
- \( x_0 = 0.175 \)
- \( x_0 = 0.3 \)
- \( x_0 = 0.5 \)
- \( x_0 = 0.75 \)

---

**Observations**

- For \( 3 < r < 4 \), \( r = 3.449 \), \( x_0 \) gets fixed, \( x_n \) is unstable, \( x_n \) will not "see" them with random initial conditions \( x_0 \).
Seems to be an infinite sequence \( r_m \), \( m = 1, 2, 3, \ldots \). If initial \( r \)-values with

\[
\lim_{m \to \infty} r_m \approx 3.569946 = r_\infty
\]

Seems to be geometric convergence \( \lim_{m \to \infty} \frac{r_m - r_{m-1}}{r_{m+1} - r_m} \approx 4.669 \) (\( \Rightarrow r_m \approx r_\infty - \cos(\frac{\pi}{m}) \))

For some, but not all, \( r_\infty < r \leq 4 \), there appears to be "chaos" (present for \( r = 4 \))

- Iterates \( x_n \) seem randomly generated even though they actually
  generated by simple formula \( x_{n+1} = r x_n (1 - x_n) \), called the
  "period-doubling route to chaos" (observed in many other models)

Is there chaos in actual populations? Not clear. Model may not be accurate enough,
unpredictably
Model may not need to be accurate; general predictions remain for large classes of models.
4. Probability

Discrete distributions

Example 4.1

Experiment: toss a die

Outcomes: 1 dot on top face, 2 dots, etc.

Sample space: set of all possible outcomes of the experiment $\Omega = \{1, 2, 3, 4, 5, 6\}$

discrete: contains no intervals

Event: any subset of $\Omega$

e.g. $A = \{2, 4, 6\}$ “getting an even number”

$B = \{1, 2\}$ “getting a number less than 3”

$A \cup B = \{1, 2, 4, 6\}$ “getting an even number or a number less than 3”

$A \cap B = \{2\}$ “getting an even number and a number less than 3”

To model a fair die, we assign equal probability ($\frac{1}{6}$) to each outcome.

Probability of an event $A$ is sum of prob. of outcomes in the event

$P(A) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$ (or $50\%$)

$P(B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

In general, let $\Omega$ be a sample space of outcomes. $\emptyset$ denotes the empty (or impossible) event. $A^c$ denotes the complementary event to $A$. (all outcomes in $\Omega$ that are not in $A$)

1. For any event $A$, $0 \leq P(A) \leq 1$

2. $P(\emptyset) = 0$, $P(\Omega) = 1$

3. If $A \cap B = \emptyset$ then $P(A \text{ or } B) = P(A) + P(B)$

4. $P(A^c) = 1 - P(A)$