Example 3.5 A discrete-time fixed iteration

\[ x_{n+1} = x_n + r - x_n^2 \]

where \( r \) is a parameter near \( 0 \).

Fixed points \( f(x) = x, \ x + r - x^2 = x, \ r - x^2 = 0 \), \( x^2 = r \). No solution if \( r < 0 \). If \( r > 0 \):

- If \( r < 0 \) there are no fixed points.
- If \( r = 0 \) there is one fixed point \( x^* = 0 \).
- If \( r > 0 \) there are two fixed points \( x_1 = -\sqrt{r}, \ x_2 = \sqrt{r} \).

Linearized stability:

\[ f(x) = x + r - x^2, \quad f'(x) = 1 - 2x. \]

- If \( r = 0 \), \( x^* = 0 \) then \( f'(0) = 1 \) and it's unstable.
- If \( r > 0 \), \( x_1 = -\sqrt{r} \) then \( f'(-\sqrt{r}) = 1 + 2\sqrt{r} > 1 \) and \( x_1 = -\sqrt{r} \) is unstable.
- If \( r > 0 \), \( x_2 = \sqrt{r} \) then \( f'(\sqrt{r}) = 1 - 2\sqrt{r} < 1 \). If \( r \) is near \( 0 \) then \( |f'| < 1 \) and \( x_2 = \sqrt{r} \) is stable.\( r < 1 \) then \(-1 < 1 - 2\sqrt{r} < 1\).

Steady-state diagrams and phase portraits:

For critical parameter value \( r = 0 \):

- \( x^* = 0 \) (unstable)

Phase portrait:

- \( x^* = 0 \) is unstable, and semistable, and therefore unstable.

Changing \( r \) from \( 0 \) translates the curve vertically by \( r \) units.

- \( r < 0 \)
- \( r > 0 \)

Bifurcation diagram (\( x \) vs \( r \), showing fixed points and stability):
Example 3.6 Discrete-time population model.

Non-overlapping generations: \( A \) is generation number \( n \geq 0, 1, 2, 3, \ldots \)

\( N_n \) is population of generation \( n \) [individuals]

Population of next generation depends on population of current generation.

\[ N_{n+1} = f(N_n) \]

\( N \) is continuous, \( n \) is discrete time

Some desirable features of model:

- \( f(0) = 0 \) and \( f(N) \) is a fixed point (no predators \( \Rightarrow \) no offspring)
- For small \( N > 0 \), \( f(N) \) should be increasing (lots of resources for individual, \( \Rightarrow \) offspring)
- For large \( N > 0 \), \( f(N) \) should be decreasing (getting crowded, \( \Rightarrow \) offspring have less chance to survive)

A particular model: discrete-time logistic model

\[ N_{n+1} = r N_n (1 - \frac{N_n}{K}) \]

where \( r, K \) are positive constants

Determine all analyzers and rescaling: non-dimensionalize the model (optimal)

Only one variable can be rescaled, time is an integer and cannot be rescaled.

Let \( x = \frac{N}{A} \), where \( A \) is a constant with units of individuals

then \( N = A \), \( N_n = A x_n \) for all \( n \), \( N_{n+1} = r N_n (1 - \frac{N_n}{K}) \) becomes

\[ A x_{n+1} = r A x_n (1 - \frac{A x_n}{K}) \]

\[ x_{n+1} = r x_n (1 - \frac{A x_n}{K}) \]

\( A \) can be chosen to eliminate only one parameter: choose \( A = K \). Then we get

\[ x_{n+1} = r x_n (1 - x_n) \]

Consider only \( 0 \leq x_n \leq 1, 0 < r \leq 4 \) so that \( 0 \leq x_{n+1} \leq 1 \), all iterates remain in \([0, 1] \) (also, maximum of \( x (1-x) \) is \( \frac{1}{4} \))

Fixed points \( f(x) = x, \) \( r x (1-x) = x \), \( r x - r x^2 = x \), \( r x^2 + (1-r) x = 0 \)

\[ x [r x + (1-r)] = 0 \]

\( x = 0 \) or \( r x + (1-r) = 0 \)

Fixed points are \( x = 0, x = \frac{1}{r} \) only in \([0, 1] \) if \( 0 < r \leq 4 \)

Plot of fixed points \( x \) vs \( r \), \( 0 \leq x \leq 1, 0 < r \leq 4 \)

If we add stability information, we get a bifurcation diagram.

Expected transcritical bifurcation at \( r_c = 1, x_c = 0 \) (intersection with oblique tangent)
Linearized stability: \( f(x) = x(x - r) = rx - rx^2, \quad f'(x) = r - 2rx \)

At \( x^* = 0 \): \( \mu = f'(0) = r \) (\( 0 < r \leq 4 \)), \( |\mu| < 1 \) for stability, i.e., \( 1 < r < 1 \)

- If \( 0 < r < 1 \) then \( f(x) \) is stable and \( x^* = 0 \) is stable.
- If \( r = 1 \) then \( f(x) \) is stable, but stability is undetermined (local bifurcation).
- If \( r > 1 \) then \( f(x) \) is unstable and \( x^* = 0 \) is unstable.