Example 3.2: Linear maps \( f(x) = \mu x \), i.e. \( x_{n+1} = \mu x_n \) or \( x \rightarrow \mu x \)

Fixed points: \( f(x) = x \) \( \Rightarrow \mu x = x \) \( \Rightarrow \mu x - x = 0 \) \( \Rightarrow (\mu - 1)x = 0 \) \( \Rightarrow x = 0 \) or \( \mu = 1 \)

- \( \mu \neq 1 \) then \( x^* = 0 \) is the only fixed point.
- If \( \mu = 1 \) then every point \( x \) is a fixed point (this map is \( x_{n+1} = x_n \)).

For linear maps, \( \mu \) is a multiplier:
\[
\begin{align*}
x_1 &= \mu x_0, \\
x_2 &= \mu x_1 = \mu^2 x_0, \\
x_3 &= \mu^3 x_0, \\
&\vdots
\end{align*}
\]

Analytically:
(a) If \( |\mu| < 1 \) then \( x_n = \mu^n x_0 \rightarrow 0 \) as \( n \rightarrow \infty \) for any initial value \( x_0 \).
(b) If \( |\mu| > 1 \) then \( |x_n| \rightarrow \infty \) as \( n \rightarrow \infty \) for any \( x_0 \), and the fixed point \( x^* = 0 \) is unstable.

Graphically:
(a) \( |\mu| < 1 \):
(i) \( 0 < \mu < 1 \) <picture> "staircase"
(ii) \(-1 < \mu < 0 \) <picture> "cobweb"
(b) \( |\mu| > 1 \):
(i) \( 1 < \mu < \infty \)
(ii) \(-\infty < \mu < -1 \)

Exercises: \( \mu = 1 \), \( \mu = -1 \), \( \mu = 0 \).

Linearized Stability:
\[
x_{n+1} = f(x_n)
\]

General (typically nonlinear) map \( f \). Suppose \( x^* \) is a fixed point (this means \( f(x^*) = x^* \)).

For stability of \( x^* \), study orbits near \( x^* \): let \( u = x - x^* \) when \( |u| \) is small.

i.e. \( x = x^* + u \)

Then \( x_n = x^* + u_n \)

\( u \) does not change with \( n \)

\[
x^* + u_{n+1} = f(x^* + u_n) \quad \text{<expanded in T. series>}
\]

\[
= f(x^*) + f'(x^*) u_n + \frac{1}{2} f''(x^*)(u_n)^2 + \ldots
\]

Linearize: ignore terms in \( u_{n+1} \), \( u_n \), ...

\[
u_{n+1} = \mu u_n \quad \text{where} \quad \mu = f'(x^*)
\]

As in Example 3.2, \( u_n = \mu^n u_0 \) so we approximate

\[
x_n \approx x^* + \mu^n (x_0 - x^*)
\]
Theorem: If |μ| < 1, then the approximation is accurate for all sufficiently close to $x^*$ and $x^*$ is stable.

If |μ| > 1, then the system is unstable.

If |μ| = 1 (i.e., $f'(x^*) = \pm 1$) then linearization does not give an accurate enough approximation to reliably predict stability, and other methods are needed (e.g., graphical).

If μ = 0 then $x^*$ is called superstable. Notice that μ = f'(x*) is the slope of the tangent line of the graph of y = f(x) where it intersects y = x.

Example 3.3 Find all fixed points of $x_{n+1} = \sin x_n$, $-\infty < x_n < \infty$, and determine the stability of each fixed point.

Fixed points: $\sin x = x$.

$x = 0$ is a solution, are there any others?

It can be proven there are no others (e.g., by proof $x - \sin x$ is increasing on $0 < x < \infty$, etc.):

Linearized stability: $f(0) = \sin x$, $f'(0) = \cos x$, $\mu = f'(0) = 1$.

Since |μ| = 1, linearized stability gives no reliable information.

Graphically we can see that $x = 0$ is stable.

Stability refers to what happens close to the fixed point.

Example 3.4 Newton's method.

To find a simple root $x^*$ of $g(x)$ ($g(x) = 0$, simple mean $g'(x^*) \neq 0$),

Newton's method:

$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$.

View this as $x_{n+1} = f(x_n)$ where $f(x) = x - \frac{g(x)}{g'(x)}$ is the "Newton's method map." $x^*$ is a root of $g$ $\iff$ $x^*$ is a fixed point of $f$. 

$g(x) = 0$ $\iff$ $f(x) = x$. 
Suppose \( g(x^*) = 0, g'(x^*) \neq 0 \). Then linearized stability of Newton’s method using

\[
\mu = f'(x^*) = 1 - \frac{g'(x^*)g''(x^*) - g(x^*)g''(x^*)}{[g'(x^*)]^2}
\]

\( \leq 0 \)  \quad \text{as long as } g''(x^*) \text{ is defined }

\(* x^* \text{ is a superstable fixed point} \)

\(* \text{Newton iterates converge to } x^* \text{ very rapidly} \)

\{
\begin{align*}
\text{for } x_n \text{ near } x^* & \text{ the number of correct decimals approximately doubles with each iteration} \\
\text{vs. increasing at a constant rate for stable but not superstable fixed point}.
\end{align*}
\}