Example 3.1. Let \( \frac{dm}{dt} + \frac{1}{2} m = 10 + 5 \sin 2t \), \( m(0) = m_0 \)

\[ m(t) = m(t, m_0) = 20 - \frac{300}{17} \cos 2t + \frac{10}{17} \sin 2t + \left( m_0 - \frac{300}{17} \right) e^{-\frac{t}{2}} \]

After one period \( T = 2 \pi/2 = \pi \) the amount of chemical in the tank is

\[ m_1 = P(m_0) = m(\pi, m_0) = \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) e^{-\frac{\pi}{2}} \]

After two periods, the amount is

\[ m_2 = m(2\pi, m_0) = \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) e^{-\frac{2\pi}{2}} = \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) \left( e^{-\frac{\pi}{2}} \right)^2 \]

etc., \( m_n = m(n\pi, m_0) = \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) \left( e^{-\frac{\pi}{2}} \right)^n \) for any \( n > 0 \).

On the other hand, let \( P(m) = \frac{300}{17} + \left( m - \frac{300}{17} \right) e^{-\frac{\pi}{2}} \). Then \( m_1 = P(m_0) \).

We compute

\[ P(m_1) = P(P(m_0)) = \frac{300}{17} + \left( P(m_0) - \frac{300}{17} \right) e^{-\frac{\pi}{2}} \]

\[ = \frac{300}{17} + \left[ \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) e^{-\frac{\pi}{2}} \right] e^{-\frac{\pi}{2}} \]

\[ = \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) \left( e^{-\frac{\pi}{2}} \right)^2 \]

\[ = m_2 \]

Similarly, \( P(m_2) = m_3 \), etc., \( m_n = P(m_n) \) for \( n = 0, 1, 2, \ldots \)

\[ m_1 = P(m_0), \quad m_2 = P(m_1) = P(P(m_0)) = P^2(m_0), \quad m_3 = P(m_2) = P(P(m_1)) = P^3(m_0) \]

\[ m_n = P^0(m_0), \quad P^n = P \text{ for } n = 2, 3, 4, \ldots, \quad P^0 = \text{Id} \]

so we can write

\[ m_n = P^n(m_0), \quad n = 0, 1, 2, \ldots \]

To find the amount of chemical in the tank after \( n \) periods, we can iterate \( P \), \( n \) times.

Notice that

\[ \lim_{n \to \infty} m_n = \lim_{n \to \infty} \left[ \frac{300}{17} + \left( m_0 - \frac{300}{17} \right) \left( e^{-\frac{\pi}{2}} \right)^n \right] = \frac{300}{17} \]

and \( P\left( \frac{300}{17} \right) = \frac{300}{17} \)

\( m^* = \frac{300}{17} \) is a fixed point of \( P \) (a solution of \( P(m) = m \)) and it is stable.

What if \( m_0 > \frac{300}{17} \)?
Graphically, a fixed point \( P(m) = m \) can be found as an intersection of the curves \( y = P(m) \) and \( y = m \). The point of intersection forms the line.

We can also graphically construct sequences \( m_0, m_1, m_2, \ldots \) and find the stability of fixed points.

One-dimensional maps

Let \( f : \mathbb{R} \to \mathbb{R} \) be a smooth function. It generates a sequence \( x_0, x_1, x_2, \ldots \) by the difference equation:

\[ x_{n+1} = f(x_n) \]

The term "map" can refer to the function \( f \), or its associated difference equation.

As seen above, \( x_n = f^n(x_0) \) where \( f^n \) is \( f \) composed \( n \) times.

The orbit (or trajectory) starting at \( x_0 \) is the sequence \( x_0, x_1, x_2, \ldots \) generated by \( f \).

Phase portraits are collections of points on a line. e.g., \( \ldots, x_{-1}, x_0, x_1, x_2, \ldots \)

A special type of orbit is a fixed point: a solution \( x = x^* \), e.g., \( f(x^*) = x^* \).

The orbit of a fixed point is itself, \( \ldots, x^*, x_1 = f(x^*), x_2, f(x_1), \ldots \).

A fixed point \( x^* \) is (asymptotically) stable if all sufficiently close \( x \) get close to \( x^* \) as \( n \to \infty \). A fixed point \( x^* \) is unstable if at least one \( x \) escapes to \( \infty \) as \( n \to \infty \).