Example 2.12 (bistability, hysteresis) cut.

Lecture in E. coli cells $\frac{dy}{dt} = y - \lambda (0.1 + \frac{y^3}{3.5})$

Bifurcation diagram: first plot equilibrium as curves in $(\lambda, y)$ plane

$0 = y - \lambda (0.1 + \frac{y^3}{3.5})$

$\lambda = \frac{0.1 + \frac{y^3}{3.5}}{y}$

(here computer to plot)

Then, add stability from phase portrait

Bistability: for $\lambda_2 < \lambda < \lambda_1$, there are two stable equilibria $y^*$

Extra, new step: Superimpose bifurcation diagram and phase portrait

Hysteresis (jumps, switching)

Imagine changing $\lambda$ very slowly (quasistatic approach $\frac{d\lambda}{dt} = 0$)

Start $\lambda < \lambda_1$, want to increase $\lambda$, new stable equilibrium $y^*$

Then increase $\lambda$ very slowly, yet it remains near $y^*$ (low-order inactivating)

"switch becomes ON"

When $\lambda$ increases just past $\lambda_1$, $y^*$ "jumps" up to near $y_1^*$ (high-order inactive con.

"switch becomes OFF"

Now decrease $\lambda$ below $\lambda_2$, $y^*$ remains near high value "switch remains ON"

Switch remains on until $\lambda$ decreases below $\lambda_1$ $y^*$

Hysteresis: "lack of reversibility" = switching occurs at different value of $\lambda$, not the same value

Fisher model of HW2 #1

$\frac{dx}{dt} = x(1-x) - \frac{x}{1+x}$

Another example, if

$x \frac{dx}{dt} = x(1-x) - \frac{x}{1+x}$

$0 < \lambda < 1$ (monotonic)

$\lambda$ is maximal harvest level, $x_h$ must be perpetual to be of fishing benefit

No conclusion

Fisher (at least in the model) maybe within real life?

Model wrong

Hysteresis may not be real

Real fisheries may be models (but not the ones in our course)
First order linear ODEs

A first order ODE is **linear** if it can be put in the form

\[ P(t) \frac{dy}{dt} + Q(t) y = G(t) , \quad P(t) \neq 0 \]

or equivalently (divide by \( P(t) \))

\[ \frac{dy}{dt} + p(t) y = g(t) \]

where \( P(t), Q(t), g(t) \) are functions of \( t \).

The linear ODE is homogeneous if \( G(t) \equiv 0 \) (i.e., \( g(t) \equiv 0 \))

\[ P(t) \frac{dy}{dt} + Q(t) y = 0 , \quad P(t) \neq 0 \quad \text{or} \quad \frac{dy}{dt} + p(t) y = 0 \]

A homogeneous ODE can be solved since it is separable

\[ \frac{dy}{y} = -p(t) dt , \quad \text{etc., etc.} \]

General solution: \( Ce^{-\int p(t) dt} \) where \( C \) is an arbitrary constant.

Especially easy for linear ODEs with constant coefficients

\[ a \frac{dy}{dt} + by = 0 , \quad a, b \text{ are} \quad \text{given}, \quad \text{with} \quad a \neq 0 \]

\[ \frac{dy}{y} = -\frac{b}{a} dt , \quad \text{gives solution} \quad \text{Ce}^{-\frac{b}{a} t} \]

**Theorem.** The general solution of the nonhomogeneous ODE

\[ \frac{dy}{dt} + p(t) y = g(t) \]

is \( y(t) = \frac{y_p(t)}{y_p(t)} + Ce^{-\int p(t) dt} \) where \( C \) is an arbitrary constant and \( y_p(t) \) is a particular solution of the nonhomogeneous ODE.

**Only specific solution.**

How can we find a particular solution? Sometimes "guess and check" (official name: method of undetermined coefficients). If the nonhomogeneous ODE has special form

\[ a \frac{dy}{dt} + by = g(t) \]

where \( a \neq 0, b \) are constant coefficients and \( g(t) \) is "nice" (a polynomial, exponential, sine, cosine),

we can guess the form of \( y_p(t) \). For example, if \( g(t) = a_0 + a_1 t + \ldots + a_n t^n \)

1. an \( n \)-th order polynomial with given coefficients \( a_0, a_1, \ldots, a_n \) Then we should guess

\[ y_p(t) = A_0 + A_1 t + \ldots + A_n t^n \]

where \( A_0, A_1, \ldots, A_n \) are undetermined coefficients.

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>( y_p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 + a_1 t + \ldots + a_n t^n )</td>
<td>( A_0 + A_1 t + \ldots + A_n t^n )</td>
</tr>
<tr>
<td>( e^{\alpha t} )</td>
<td>( A e^{\alpha t} ) if ( \alpha = -\frac{b}{a} ), ( A e^{-\frac{b}{a} t} ) if ( \alpha = -\frac{b}{a} )</td>
</tr>
<tr>
<td>( \cos \beta t ) or ( \sin \beta t ) (or both)</td>
<td>( A \cos \beta t + B \sin \beta t )</td>
</tr>
</tbody>
</table>
Example 2.13  See Example 2.1  Sep 17-22

\[ m \frac{dv}{dt} = mg - yv \]

Write as \[ m \frac{dv}{dt} + yv = mg \]

Homogeneous linear eqn. \[ m \frac{dv}{dt} + yv = 0 \] (equivalently, \[ \frac{dv}{dt} = -\frac{y}{m}v \])

Homogeneous soln. \[ C e^{-\frac{y}{m}t} \]

Particular soln. \[ g(t) = mg \] is a 0-degree polynomial (i.e., a constant), guess particular soln. \[ v_p(t) = A \] a 0-degree polynomial with undetermined coeff. \( A \)

Sub into nonhomog. eqn. to determine \( A \)

\[ m \frac{dv}{dt} + yv = mg \]

\[ yA = mg \quad A = \frac{mg}{y} \]

By Theorem, general soln. of \[ m \frac{dv}{dt} = mg - yv \] is \[ v(t) = \frac{mg}{y} + Ce^{-\frac{y}{m}t}, \text{Const.} \]