Linearized stability

Computational way of predicting stability (usually)

Autonomous ODE \( \frac{dy}{dt} = f(y) \)

Solution \( y^* \) is an equilibrium. What happens to small perturbations from \( y^* \)?

\[
y(t) = y^* + ut
\]

\( u \) is a perturbation to the solution \( y^* \)

Taylor's Theorem

\[
\frac{dy}{dt} = f(y) = f(y^*) + f'(y^*)u + \frac{1}{2} f''(y^*)u^2 + \ldots
\]

Linearize: keep only up to linear terms in \( u \)

\[
\frac{dy}{dt} = \lambda u
\]

where \( \lambda = f'(y^*) \)

This approximation is the solution to perturbation \( u(t) \). Solve with \( Ce^{\lambda t} \)

\[
y(t) = y^* + Ce^{\lambda t}
\]

Then: \( f'(y^*) = \lambda < 0 \) (exp. decay) \( \Rightarrow y^* \) is asymptotically stable

Then: \( f'(y^*) = \lambda > 0 \) (exp. growth) \( \Rightarrow y^* \) is unstable

\( f'(y^*) = \lambda = 0 \) (lin. stable)

Not a good approximation is not a good enough approximation.

\( f'(y^*) \) is slope of graph \( \frac{dy}{dt} \) or y curve at intersection

\[
\begin{align*}
\text{ex. i) } & \quad \frac{dx}{dt} = x(1-x) \\
\text{ex. ii) } & \quad \frac{dx}{dt} = x(1-2x) \\
\text{ex. iii) } & \quad \frac{dy}{dt} = -y^2
\end{align*}
\]

Graphically:

Linearized stability:

\( \lambda = f'(y^*) = 1 < 0 \) \( \Rightarrow \) lin. stable

\( x^* = 0 \) is stable

Linearized stability:

\( \lambda = f'(y^*) = 1 > 0 \) \( \Rightarrow \) lin. unstable

\( x^* = 0 \) is unstable

\( \text{Graphically: } \frac{dy}{dt} = -y^2 \)
Example 2.2 with $x$

Logistic population growth, dimensionless version

Explicit solution:

$$\frac{dx}{dt} = (1-x) x$$

Separate variables:

$$\frac{dx}{(1-x) x} = dt$$

Integrate:

$$\int \frac{dx}{(1-x) x} = \int dt$$

$$\frac{1}{(1-x) x} = \frac{a}{1-x} + \frac{b}{x}$$

Substitute $a = -b$:

$$\frac{dx}{(1-x) x} = \frac{dx}{(1-x) x}$$

$$\int \frac{dx}{1-x} = -\ln|1-x| + \ln|x| = \ln\left|\frac{x}{1-x}\right|$$

$$\ln\left|\frac{x}{1-x}\right| = t + C$$

$$\frac{x}{1-x} = e^{t+C} = Ce^t$$

$$x = Ce^t (1-x), \quad x = Ce^t - Ce^t x$$

$$x = \frac{Ce^t}{1+Ce^t} = \frac{x_0 e^t}{x_0 + (1-x_0)e^t}$$

Where $t \geq 0$, $x(0): x_0$:

$$x(t) = \frac{x_0}{1-x_0} e^{rt}$$

Solution:

$$C = \frac{x_0}{1-x_0} e^{rt}$$

$$x(t) = \frac{x_0}{1-x_0} e^{rt}$$

$$x(t) = \frac{x_0 e^{rt}}{1+e^{rt}}$$

$$x(t) = \frac{x_0}{1-x_0}$$

Exercise. Find explicit solution of $\frac{dx}{dt} = k(t) N, \quad y(t), y_0$.

Bifurcations

Most models have parameters. Sometimes parameters do not change qualitative features of solution. (E.g. $m, g$ in $\frac{dx}{dt} = -g v + r t$ or $r, k$ in $\frac{dx}{dt} = r N (1 - \frac{N}{K})$.)

But, sometimes, changing parameters can change qualitative features of solutions (phase portraits)
Example 2.3 \( \frac{dy}{dt} = \mu - y^2 \) where \( \mu \) is a (constant) parameter.

Graphically plot \( \frac{dy}{dt} \) vs. \( y \) for different values of \( \mu \).

\( \mu < 0 \)

Phase portrait (\( y \approx 0 \))

\( \mu > 0 \)

For \( \mu \) arbitrarily close to \( \mu = 0 \) there are no different phase portraits.

We say a bifurcation occurs at \( \mu = 0 \).

Solve for equilibrium:

\( 0 = \mu - y^2 \)

\( y^2 = \mu \) \( y = \pm \sqrt{\mu} \) but only if \( \mu > 0 \)

\( \mu < 0 \) no equilibrium \( \mu > 0 \) one equilibrium \( y = 0 \)

\( \mu > 0 \) two equilibrium \( y = -\sqrt{\mu} \), \( y = +\sqrt{\mu} \)