Vector field \( \frac{dy}{dt} = -0.2v + 9.8 \) on \( uv\)-axis.

Alternate representation: plot \( \frac{dy}{dt} = v \) vs \( u \)

\( y \)-vector field (preferred)

Phase portrait

\( \frac{dy}{dt} = -0.2v + 9.8 \)

i) \( \frac{dy}{dt} = 0 \) equilibrium \( v = 49 \)

ii) \( v < 49 \): \( \frac{dy}{dt} < 0 \), \( v(t) \) decreasing (move to right)

iii) \( v > 49 \): \( \frac{dy}{dt} > 0 \), \( v(t) \) increasing (move to left)

Equilibrium is (geographically) stable: small perturbations from equilibrium will evolve back toward equilibrium

Relationship between geometric interpretations 1 and 2 (only for autonomous 1st order ODEs)

Vector field \( \rightarrow \) integral curve

integral curve \( \leftrightarrow \) trajectory \( (v(t)) \) only

Dimensional analysis and scaling

Make ODE "look simpler" by "removing" parameters.

Define dimensionless variables \( \frac{v}{A} \) where \( A \) is a constant with units of velocity.\( \frac{dy}{dt} = -\frac{V}{A}v + g \)

New variables \( x = A \) where \( B \) is a constant with units of time e.g. seconds, \( v, t \) have no units

- New no units dimensionless "moving" variables

Change constants \( (A, B) \) to dimensionless eq. looks nice.

Change \( y \) to \( \frac{dy}{m} = 1 \); \( B = \frac{mg}{m} = g \)

Change \( \frac{v}{A} \) to \( \frac{1}{A} \); \( \frac{A}{B}g = \frac{mg}{m} = g \)

Summary: let \( w = \frac{v}{A} \), \( t = \frac{m}{B} \). Then \( \frac{dw}{dt} = -\frac{V}{A}v + g \) becomes

\( \frac{dw}{dt} = -w + 1 \)
Equilibria and stability

Autonomous ODE \( \frac{dy}{dt} = f(y) \)

An equilibrium is a solution \( y^*, \) of \( f(y) = 0 \)

An equilibrium \( y^* \) is \((\text{uniquely})\) \textit{stable} if all sufficiently small perturbations from \( y^* \)
give solutions \( y(t) \) that stay close to \( y^* \) for all \( t \geq 0 \) and approach \( y^* \) as \( t \to +\infty \)
An equilibrium \( y^* \) is \textit{unstable} if at least some arbitrarily small perturbations from \( y^* \) give solutions \( y(t) \) that do not remain sufficiently close to \( y^* \) for all \( t \geq 0 \).

Population growth \( \underline{\text{Example 2.2 Logistic growth model}} \)

\( N(t) \) = population at time \( t \), number of individuals

\( \underline{\text{(approximately with continuous function, reasonable if population is large)}} \)

Simplest model \( \frac{dN}{dt} = rN \) \( \text{(exponential growth)} \)

Per capita growth rate \( \frac{dN}{N} = \frac{r}{N} \) \text{ positive constant (parameter)} \( \left[ \frac{\text{individuals}}{\text{time}} \right] = \left[ \frac{\text{time}^{-1}}{\text{individuals}} \right] \)

More realistic model: \( \text{per capita growth rate is a decreasing function of } N \)

\( \frac{dN}{dt} = g(N) \) \text{ some decreasing function of } N \( \text{ (due to competition for finite resources)} \)

\( \underline{\text{logistic model}} \quad g(N) = r \left( 1 - \frac{N}{K} \right) \text{ positive constant [individuals]} \quad \left[ \text{time}^{-1} \right] \text{ \text{intrinsic}} \) \text{ per capita growth rate [time]}

\( \underline{\text{Logistic ODE}} \quad \frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N \quad (N \geq 0 \text{ for populations}) \)

Write this in \underline{\text{dimensionless form}}:

\( \underline{\text{dimensionless population}} \quad x = \frac{N}{A} \)

\( \underline{\text{dimensionless time}} \quad t = \frac{t}{B} \)

As in previous model: \( N = Ax, \quad \frac{d}{dt} = \frac{1}{B} \frac{dx}{dt} \)

\( \frac{d}{dt} N = r \left( 1 - \frac{N}{K} \right) N \)

Becomes

\( \frac{d}{dt} Ax = r \left( 1 - \frac{Ax}{K} \right) Ax \)

\( \frac{dx}{dt} = rB \left( 1 - \frac{A}{K} x \right) x \)

Choose \( A, B \) so that \( rB = 1, \quad \frac{A}{K} = 1 \)

\( B = \frac{1}{r \left[ \text{time}^{-1} \right]} = \left[ \text{time} \right], \quad A = K \left[ \text{individuals} \right], \)

Summary: let \( x(t) = \frac{N(t)}{K}, \quad t = rt \). Then \( \frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N \) hence \( \frac{dx}{dt} = (1-x)x \)
Vector field \( \frac{dx}{dt} = x \)

Equilibrium: \( D = (1-x)x \); \( x^* = 0, 1 \)

Phase portrait

Induced curves in \( xt \)-plane (guess)

In original variables
\[ x = \frac{N}{K}, \quad t = \frac{Kx}{x^*} \]
\( x^* = 0 \) corresponds to \( N^* = 0 \), \( x^* = 1 \) corresponds to \( N^* = K \)

\( t = 1 \) corresponds to \( t = \frac{K}{x^*} \) etc.