\[ s(N) = \text{probability of one offspring surviving to maturity, from a clutch size of } N \]

For simplicity, assume \( s(N) \) is a linear function

\[ s(N) = 1 - eN \]

where \( e > 0 \) is a constant parameter that could vary with species, habitat, etc.

Fitness = number of offspring that reach maturity, per parent

\[ = (\text{no. of eggs})(\text{survival probability}) \]

\[ = N \cdot s(N) \quad \text{< could divide by 2, won't change result> } \]

\[ = N(1 - eN) \]

\[ f(N) = N - eN^2 \]

Model problem: find the maximum value of \( f(N) = N - eN^2 \), \( 0 \leq N \leq \frac{1}{2} \).

Straightforward: graph is parabola, opening downward, \( f(N) = 0 \) if \( N = 0, \frac{1}{2c} \).

Find critical point: \( f'(N) = 1 - 2cN = 0 \quad \Rightarrow N = \frac{1}{2c} \) \quad check it's in domain \( f''(N) = -2c < 0 \) \quad verifies that is local max by 2nd deriv. test.

Since graph is parabola, \( N = \frac{1}{2c} \) is absolute max.

Indefinite result:

For maximum fitness, clutch size should be \( \frac{1}{2c} \).

Actual clutch size is an open question:

Think about prediction: expect populations to evolve so clutch sizes are nearly optimal: go out in field and measure sizes (many times) take average, compare to \( \frac{1}{2c} \) \quad requires measurement of \( e \). (Indefinitive experiment)

(Don't get emotionally attached to a model: if it fails to usefully predict reality, use a better model.)
Example 1.3 Optimal foraging time.

Where should you stop foraging?

Imagine going out to forage for food. Travel time $T_0 > 0$.

Once time, return is a decreasing function of time:

\[
\frac{\text{return}}{\text{time spent foraging}} = \frac{r(t)}{T_0 + t}
\]

Where is the best time to stop foraging and move on to some other location?

Optimize quantity $\frac{\text{return}}{\text{time used}} = \frac{r(t)}{T_0 + t}$

**Mathematical interpretation**

Maximize $\frac{r(t)}{T_0 + t}$ (true picture, some economics should exist)

Find critical point:

\[
f'(t) = \frac{r'(t)(T_0 + t) - r(t) \cdot 1}{(T_0 + t)^2}
\]

\[
f'(t) = 0 \implies r'(t) = \frac{r(t)}{T_0 + t} = f(t)
\]

Critical point occurs when tangent to $r(t)$ has same slope as $\frac{r(t)}{T_0 + t}$.

Verify this is at least a local max.

\[
f''(t) = \frac{r''(t)(T_0 + t) + r'(t) - r(t)}{(T_0 + t)^2} \cdot \left[ \frac{r'(t)(T_0 + t) - r(t)}{(T_0 + t)^2} \right] + \frac{r''(t)(T_0 + t) - r(t)}{(T_0 + t)^2} \cdot 2 \frac{(T_0 + t)}{(T_0 + t)^2}
\]

\[
f''(t) = r''(t) \cdot \frac{1}{T_0 + t} < 0 \quad \text{if} \quad r(t) \text{ is always concave down}
\]

So $f(t^*)$ is a local max value.

For explicit value $t^*$, need explicit form for $r(t)$.

There are various possibilities for $r(t)$ e.g. $r(t) = \frac{at}{t+b}$ where $a > 0, b > 0$ are positive (constants)

\[
r'(t) = \frac{a(t+b) - at}{(t+b)^2} = \frac{ab}{(t+b)^2}
\]

\[
r'(t) = \frac{r(t)}{T_0 + t} \implies \frac{ab}{(t+b)^2} = \frac{r(t)}{T_0 + t} \implies \frac{b}{t+b} = \frac{T_0}{t+T_0}
\]

\[
f(t+T_0) = t(t+b)
\]

\[
bT_0 = t^2 \quad t^2 = \sqrt{bT_0}
\]
2. **Continuous-Time Dynamical Systems** in One Dimension

Models of the form \( \frac{dx}{dt} = f(x) \) in one dimension.

**Exponential Growth and Decay**

- Population growth: \( \frac{dN}{dt} = rN \) (size of population \( N(t) \))
- Radioactive decay: \( \frac{dm}{dt} = -\lambda m \) (mass of radioactive material \( m(t) \))
- Newton's law of cooling: \( \frac{dT}{dt} = k(T - T_e) \) (temperature of object \( T(t) \))

All are models of the type

\( \frac{dy}{dt} = ay + b \) where \( a, b \) are constants (parameters), \( a \neq 0 \)

**Differential Equation (DE):** eqn. involving one or unknown function or one or more of its derivatives